

Consistent truncations of M-theory for general $SU(2)$ structures

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ABSTRACT

In seven dimensions any spin manifold admits an $SU(2)$ structure and therefore very general M-theory compactifications have the potential to allow for a reduction to $N = 4$ gauged supergravity. We perform this general $SU(2)$ reduction and give the relation of $SU(2)$ torsion classes and fluxes to gaugings in the $N = 4$ theory. We furthermore show explicitly that this reduction is a consistent truncation of the eleven-dimensional theory, in other words classical solutions of the reduced theory also solve the eleven-dimensional equations of motion. This reduction generalizes previous M-theory reductions on Tri-Sasakian manifolds and type IIA reductions on Calabi-Yau manifolds of vanishing Euler number. Moreover, it can also be applied to compactifications on certain G_2 holonomy manifolds and to more general flux backgrounds.

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1 Introduction

It has been argued that the vanishing of certain topological indices of the compactification manifold restricts the appearance of string corrections due to the appearance of additional (spontaneously broken) supercurrent [1]. This was exemplified in [1] for Calabi-Yau compactifications of type II for the case of vanishing Euler number. The vanishing Euler number ensures the existence of an $SU(2)$ structure on the Calabi-Yau manifold. Reduction on such an $SU(2)$ structure leads to $N = 4$ gauged supergravities [1–6], which can be conveniently described by the embedding tensor formalism [7, 8].

Non-renormalization theorems of $N = 4$ supergravity then explain the vanishing of perturbative string corrections for these compactifications and lead to the conjecture that also certain non-perturbative string corrections must vanish for these backgrounds. In the discussed case of Calabi-Yau manifolds with vanishing Euler number it could be shown by applying mirror symmetry that this conjecture indeed is true.

These findings suggest that spontaneously broken supercurrents play a far more important role in string compactifications than considerations of effective actions would suggest. In particular this suggests that there is a general scheme to understand string corrections for general G-structure backgrounds. The most pressing question is whether spontaneously broken supercurrents can also restrict string corrections when only $N = 1$ remains unbroken.

A particularly interesting case to address this question are M-theory compactifications to four dimensions. It has been known for a long time that any seven-dimensional spin manifold admits an $SU(2)$ structure [9–11]. This suggests that we should be able to find for many such M-theory compactifications a reduction to $N = 4$ gauged supergravity, which might give strong constraints on membrane instanton corrections in these backgrounds. In this paper we will perform such an $SU(2)$ structure reduction to four dimensions and determine the corresponding $N = 4$ gauged supergravity by identifying the corresponding gaugings.

We show that the reduction performed in this paper is in fact a consistent truncation. Consistent truncations to gauged supergravities have been performed for many particular AdS backgrounds, see for instance [12–18]. Our reduction generalizes the known M-theory reductions to $N = 4$ gauged supergravity. Therefore it might help to understand more general four-dimensional AdS backgrounds.

The given reduction is applicable both to compactifications to Minkowski and AdS spacetimes. A particularly interesting application would be to understand the corrections to M-theory compactifications on G_2 manifolds: Some of the known Joyce manifolds of [19] are dual heterotic Calabi-Yau backgrounds with vanishing Euler number [20], where the techniques of [1] could be used.

The paper is organized as follows. In Section 2 we discuss general $SU(2)$ structure manifolds in seven dimensions and thereby set the stage for performing the reduction. In Section 3 we will make the reduction ansatz and then perform the $SU(2)$ reduction to four dimensions. The embedding tensor components of the corresponding $N = 4$ gauged supergravity are identified in Section 4. In Section 5 we discuss the consistency of the truncation, and in Section 6 we make contact with some classes of AdS vacua in the literature. Some of the technical details as well as our conventions regarding $N = 4$ gauged supergravity are presented in three appendices.

2 $SU(2)$ structures on seven-manifolds

Let us start by introducing the concept of an $SU(2)$ structure on a seven-dimensional manifold Y . On a seven-dimensional manifold the spinor bundle is eight-dimensional. We will be interested in the splitting $SO(7) \rightarrow SO(3) \times SO(4)$ which reads for the corresponding spin groups $\text{Spin}(7) \rightarrow SU(2) \times SU(2) \times SU(2)$. A seven-dimensional

$SU(2)$ structure manifold Y admits four nowhere vanishing spinors η_{ii} , $i, \hat{i} = 1, 2$, whose norm we fix by imposing

$$\bar{\eta}^{j\hat{j}}\eta_{ii} = \delta_i^j \delta_{\hat{i}}^{\hat{j}} . \quad (2.1)$$

These four spinors arise under the breaking $\text{Spin}(7) \rightarrow SU(2)_3 \times SU(2)_4 \times SU(2) \rightarrow SU(2)$ by

$$(8) \rightarrow (2, 2, 1) \oplus (2, 1, 2) \rightarrow 4(1) \oplus 2(2) . \quad (2.2)$$

We denoted in (2.1) the index of the broken $SU(2)_3$ that is related to the $SO(3)$ by i and the index of the broken $SU(2)_4$ inside $SO(4)$ by \hat{i} . The third $SU(2)$ subgroup is the (unbroken) structure group.

Based on these spinors we can introduce a $SU(2)$ triple of (real) two-forms $J^{\hat{a}}$, $\hat{a} = 1, 2, 3$, and a triple of (real) one-forms K^a , $a = 1, 2, 3$, via

$$J^{\hat{a}} = i \sqrt{\frac{3}{2}} (\sigma^{\hat{a}})^i_j \bar{\eta}^{i\hat{j}} \gamma_{mn} \eta_{ii} dx^m \wedge dx^n , \quad K^a = (\sigma^a)^i_j \bar{\eta}^{j\hat{i}} \gamma_m \eta_{ii} dx^m . \quad (2.3)$$

The Fierz identities for these spinors can now be parametrized by

$$\begin{aligned} \eta_{ii} \otimes \bar{\eta}^{j\hat{j}} = & \frac{1}{8} \left((1 + \gamma_{(3)}) \delta_i^j + i(\sigma^a)^i_j (K_m^a \gamma^m - \epsilon^{abc} K_m^b K_n^c \gamma^{mn}) \right. \\ & \left. \cdot \left((1 - \gamma_{(4)}) \delta_{\hat{i}}^{\hat{j}} + \sqrt{\frac{2}{3}} i(\sigma_{\hat{a}})^{\hat{j}}_{\hat{i}} J_{pq}^{\hat{a}} \gamma^{pq} \right) \right) . \end{aligned} \quad (2.4)$$

Taking the products of these bilinears and using (2.1) yields the relations

$$\frac{1}{2} J^a \wedge J^b = \delta^{ab} \text{vol}_4 \quad (2.5)$$

and

$$\epsilon^{abc} K^a \wedge K^b \wedge K^c = \text{vol}_3 . \quad (2.6)$$

The Fierz identities guarantee also the existence of an almost product structure $P : TY \rightarrow TY$, $P^2 = id$, on the manifold, defined locally via

$$P_m{}^n = 2K_m^a \hat{K}_a^n - \delta_m^n . \quad (2.7)$$

where the vectors \hat{K}_a are defined by

$$K^a(\hat{K}_b) = \delta_b^a , \quad J^{\hat{a}}(\hat{K}^b, \cdot) = 0 . \quad (2.8)$$

The eigenspaces T_3Y and T_4Y of P to the eigenvalues $+1$ and -1 respectively yield a global decomposition of the tangent space,

$$TY = T_3Y \oplus T_4Y . \quad (2.9)$$

The subbundle T_3Y is trivial, spanned by \hat{K}_a . By definition of P , the K^a ($J^{\hat{a}}$) are trivial on T_4Y (T_3Y). Note that the splitting (2.9) and the definition of such an $SU(2)_3$ triple of one-form K^a and an $SU(2)_4$ triple of two-forms $J^{\hat{a}}$, $\hat{a} = 1, 2, 3$, which satisfy the conditions (2.5) and (2.6), allow for a definition of the $SU(2)$ -structure with no reference to spinors.

Let us now discuss the frame bundle over spacetime times Y . We choose a section (vielbein)¹

$$e^A = (e^\mu, K^a, e^\alpha) , \quad (2.10)$$

¹Eq. (2.6) implies that the K^a can be chosen as components of the vielbein.

where the e^μ live in spacetime and depend only on spacetime coordinates. In contrast, the $K^a = k_b^a(v^b + G^a)$ consist of one-forms v^a in T_3^* and spacetime gauge fields G^a (the Kaluza-Klein vectors) that parameterize the fibration of T_3^* over spacetime, as well as the coefficient k_b^a , which is a spacetime scalar.² Furthermore, the e^α are one-forms on T_4^* such that

$$J^{\hat{a}} = \frac{1}{2}(I^{\hat{a}})_{\beta}^{\alpha} e^{\alpha} \wedge e^{\beta} , \quad (2.11)$$

with constant coefficients $(I^{\hat{a}})_{\beta}^{\alpha}$ that are the generators of the $SU(2)_4$ algebra of complex structures on the frame bundle, i.e.

$$(I^{\hat{a}})_{\gamma}^{\alpha} (I^{\hat{b}})_{\beta}^{\gamma} = \epsilon^{\hat{a}\hat{b}\hat{c}} (I^{\hat{c}})_{\beta}^{\alpha} - \delta^{\hat{a}\hat{b}} \delta_{\beta}^{\alpha} . \quad (2.12)$$

Similarly, the \tilde{I}^a are generators of $SO(3)$ given by $(\tilde{I}^a)_c^b = \epsilon^{abc}$. The dual vielbein to (2.10) is

$$\hat{e}_A = (\hat{e}_{\mu}, \hat{K}_a, \hat{e}_{\alpha}) = (\partial_{\mu} - G_{\mu}^a \hat{v}_a, (k^{-1})_a^b \hat{v}_b, \hat{e}_{\alpha}) , \quad (2.13)$$

where \hat{v}_a are the vector fields dual in T_3^* to the vielbein component v^a .

Next, we consider the Levi-Civita connection one-form Ω , which is the unique torsion-free connection satisfying the Maurer-Cartan equation

$$De = de + \Omega \wedge e = 0 . \quad (2.14)$$

The corresponding curvature two-form is defined by

$$R = d\Omega + \Omega \wedge \Omega . \quad (2.15)$$

The Ricci tensor (in flat indices) is defined by contraction with the dual vielbein,

$$\text{Ric}_{AB} = R_A^C (\hat{e}_C, \hat{e}_B) , \quad (2.16)$$

and the Ricci scalar as its trace

$$r_{11} = \text{Ric}_{AB} \delta^{AB} . \quad (2.17)$$

Let us decompose the eleven-dimensional connection under the breaking of the Lorentz group $SO(1, 10) \rightarrow SO(1, 3) \times SO(3) \times SO(4)$ as

$$\begin{aligned} \mathbf{55} &= (\mathbf{6}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{4}, \mathbf{1}, \mathbf{4}) \oplus (\mathbf{4}, \mathbf{3}, \mathbf{1}) \oplus ((\mathbf{1}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{4}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{6})) , \\ \Omega &= \omega + [\lambda] + [\gamma] + \Theta , \end{aligned} \quad (2.18)$$

where we have called the full $SO(7)$ connection Θ . Using $SO(3) \equiv (SU(2)_3)/\mathbb{Z}_2$ and $SO(4) \equiv (SU(2)_4 \times SU(2))/\mathbb{Z}_2$, we can further decompose the adjoint representation of $SO(7)$ and thus the internal connection Θ as

$$\begin{aligned} so(7) &= su(2)_3 \oplus su(2) \oplus su(2)_4 \oplus (\mathbf{3}, \mathbf{2}, \mathbf{2}) , \\ \Theta &= [\phi^a] + \theta + [\psi^{\hat{a}}] + [\tau] , \end{aligned} \quad (2.19)$$

²Due to the mixed spacetime/internal components of the ten-dimensional metric, the components K^a of the vielbein are not purely internal. We will nevertheless retain the same nomenclature as in (2.3) for simplicity.

where $su(2)$ is the adjoint of the $SU(2)$ structure group, $su(2)_4$ is spanned by the $I^{\hat{a}}$ and $su(2)_3$ by $(I^a)_c^b = \epsilon^{abc}$. Using this decomposition and the vielbein (2.10), the Maurer-Cartan equations (2.14) read in components

$$\begin{aligned} de^\mu + \omega_\nu^\mu \wedge e^\nu + \lambda_\alpha^\mu \wedge e^\alpha + \gamma_a^\mu \wedge K^a &= 0 , \\ dK^a + \epsilon^{abc} \phi^c \wedge K^b + \tau_\alpha^a \wedge e^\alpha + \gamma_\mu^a \wedge e^\mu &= 0 , \\ de^\alpha + \theta_\beta^\alpha \wedge e^\beta + (I^{\hat{a}})_\beta^\alpha \psi^{\hat{a}} \wedge e^\beta + \tau_a^\alpha \wedge K^a + \lambda_\mu^\alpha \wedge e^\mu &= 0 . \end{aligned} \quad (2.20)$$

Note that the connection component θ is the torsionful $SU(2)$ connection. Its internal torsion two-form T can be expressed in terms of the other components of Θ . On T_3Y the internal torsion is given by $T^a = dK^a$ and the component on T_4 is

$$T^\alpha = de^\alpha + \theta_\beta^\alpha \wedge e^\beta = -(I^{\hat{a}})_\beta^\alpha \phi^{\hat{a}} \wedge e^\beta - \tau_a^\alpha \wedge K^a . \quad (2.21)$$

Similar to the connection one-form we can also decompose the Ricci tensor group-theoretically. In particular, we are interested in the ‘symmetric’ representation S^2T^*Y , which decomposes as

$$\begin{aligned} S^2T^*Y &= S_0^2T_3^*Y \oplus \mathbb{R}g_3^{(0)} \oplus S_0^2T_4^*Y \oplus \mathbb{R}g_4^{(0)} \oplus (T_3^*Y \otimes T_4^*Y) \\ &= (\mathbf{5}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}, \mathbf{2}) . \end{aligned} \quad (2.22)$$

Here, the $(\mathbf{1}, \mathbf{3}, \mathbf{3})$ representation $S_0^2T_4^*$ is spanned by the products of generators of $su(2)_4$ and $su(2)$. In other words, since the elements of $su(2)$ and $su(2)_4$ commute, the representation can be written as

$$S_0^2T_4^* = \{(I^{\hat{a}})_\gamma^\alpha (\mathcal{I}_{\hat{a}})_\beta^\gamma | \mathcal{I}_{\hat{a}} \in su(2), \hat{a} = 1, 2, 3\} . \quad (2.23)$$

3 Dimensional reduction from M-theory

In this section, we will reduce the eleven-dimensional supergravity action

$$\begin{aligned} S_{11} &= \frac{1}{2\kappa_{11}^2} \int_{11} ((*_11 1)r_{11} - \frac{1}{2}G_4 \wedge *_11 G_4) \\ &\quad - \frac{1}{6} \int_{11} G_4 \wedge G_4 \wedge C_3 , \end{aligned} \quad (3.1)$$

to four dimensions. Here, $G_4 = dC_3$ is the form field strength of the three-form gauge field.

3.1 The reduction ansatz

The almost product structure (2.7) on Y will play a central role in the choice of our reduction ansatz. T_3 has trivial structure group and is therefore parallelizable. We hence introduce a basis of three global one-forms v^a , $a = 1, 2, 3$, on this subbundle, yielding three one-forms, three two- and a three-form (their wedge products) as expansion forms.

On T_4 our ansatz similarly to [1] contains $SU(2)$ singlets and triplets. It is easily checked that $SU(2)$ doublets exactly correspond to odd forms on T_4 . Therefore, the ansatz will consist of two-forms $\omega^I, I = 1, \dots, n$, that all square to the same volume form $\text{vol}_4^{(0)}$ on T_4 , i.e.

$$\frac{1}{2}\omega^I \wedge \omega^J = \eta^{IJ} \text{vol}_4^{(0)}, \quad (3.2)$$

where η is a metric with signature $(3, n-3)$, reflecting the number of singlet and triplet representations as discussed above. Furthermore, we include all wedge products of ω^I and v^a in the reduction ansatz. For instance, we expand the forms $J^{\hat{a}}$ and K^a of (2.3) that specify the $SU(2)$ structure in the set of modes $\omega^I, I = 1, \dots, n$, and $v^a, a = 1, 2, 3$, i.e.

$$J^{\hat{a}} = e^{\rho_4/2} \zeta_I^{\hat{a}} \omega^I, \quad K^a = e^{\rho_3/3} k_b^a (v^b + G^b), \quad (3.3)$$

where $\det(k) = 1$. Furthermore, we fix $\epsilon^{abc} v^a \wedge v^b \wedge v^c = \text{vol}_3^{(0)}$. Note that a consequence of the second equation in (3.3) is that

$$k_d^a k_e^b k_f^c \epsilon^{def} = \epsilon^{abc}, \quad k_d^a k_e^b k_f^c \epsilon_{abc} = \epsilon_{def}, \quad (3.4)$$

which in particular means that $\det(k) = 6$.

Note that the presence of internal one-forms in our ansatz gives rise to Kaluza-Klein vectors G^i , i.e. mixed spacetime and internal components of the ten-dimensional metric. The expansion coefficients $\zeta_I^{\hat{a}}, \rho_4, \rho_3$ and k_b^a depend on the spacetime coordinates and give rise to scalar fields in four dimensions. Furthermore, (2.5) yields the relations

$$\zeta_I^{\hat{a}} \eta^{IJ} \zeta_J^{\hat{b}} = \delta^{\hat{a}\hat{b}}, \quad (3.5)$$

and

$$\text{vol}_4 = e^{\rho_4} \text{vol}_4^{(0)}, \quad \text{vol}_3 = e^{\rho_3} \text{vol}_3^{(0)}. \quad (3.6)$$

The four-dimensional fields $\rho_{3/4}$ describe the volume moduli of $T_{3/4}$ while the $\zeta_I^{\hat{a}}$ describe the $SU(2)$ -structure geometry and k_b^a describes the three-dimensional geometry.

We can also expand the three-form gauge field in terms of this basis. This gives

$$\begin{aligned} C_3 = & \hat{C} + \hat{C}_a \wedge (v^a + G^a) + C_I \wedge \omega^I + \frac{1}{2} \epsilon_{abc} C^a \wedge (v^b + G^b) \wedge (v^c + G^c) \\ & + \frac{1}{6} c_0 \epsilon_{abc} (v^a + G^a) \wedge (v^b + G^b) \wedge (v^c + G^c) + c_{aI} (v^a + G^a) \wedge \omega^I. \end{aligned} \quad (3.7)$$

We will also describe fluxes in this setup, therefore our ansatz for the four-form field strength will be

$$G_4 = G_4^{\text{flux}} + dC_3, \quad (3.8)$$

where we define

$$G_4^{\text{flux}} = e^{-3\phi - \rho_3} f_0 \text{vol}_4(\text{Mink}) + g_0 \text{vol}_4^{(0)} + \frac{1}{2} g_I^a \epsilon_{abc} v^b \wedge v^c \wedge \omega^I, \quad (3.9)$$

with f_0, g_0 and g_I^a being constants, and we demand the Bianchi identity

$$dG_4 = 0. \quad (3.10)$$

Furthermore, G_4^{flux} is only defined up to an exact piece, so that only a subset of the numbers (g_0, g_{aI}) are actual flux numbers. Also, note that the flux piece in (3.9) proportional

to $\text{vol}_4(\text{Mink})$ has a dependence on the volume factors because it originates from dual seven-form flux

$$G_7^{\text{flux}} = f_0 \text{vol}_3^{(0)} \wedge \text{vol}_4^{(0)} . \quad (3.11)$$

Note that the flux piece in (3.9) proportional to $\text{vol}_4(\text{Mink})$ can be absorbed in $d\hat{C}$ but will reoccur later when we introduce dual fields. We discuss this seven-form flux again in Appendix B.1.

The $J^{\hat{a}}$ and the K^a in general define the Hodge star, which splits into a spacetime component and two components $*_3$ and $*_4$ acting on forms on T_3 and T_4 , respectively. The precise form of $*_3$ and $*_4$ is fixed by

$$*_3 K^a = \frac{1}{2} \epsilon_{abc} K^b \wedge K^c , \quad *_3 1 = \text{vol}_3 , \quad *_4 J^{\hat{a}} = J^{\hat{a}} . \quad (3.12)$$

The latter can be translated into

$$*_4 \omega^I = (2\zeta^{aI} \zeta_J^a - \delta_J^I) \omega^J = H_J^I \omega^J , \quad *_4 1 = \text{vol}_4 = e^{\rho_4} \text{vol}_4^{(0)} . \quad (3.13)$$

In the following reduction, we will assume that the internal volume is normalized,

$$\int_7 \frac{1}{6} \epsilon^{abc} v^a \wedge v^b \wedge v^c \wedge \text{vol}_4^{(0)} = \frac{\kappa_{11}^2}{\kappa_4^2} . \quad (3.14)$$

To perform the reduction, we must next specify the differentials of the expansion forms $\{v^i, \omega^I\}$. As remarked above, we will require that the differential algebra of modes they span closes, i.e.

$$\begin{aligned} dv^a &= \frac{1}{2} t^{ab} \epsilon_{bcd} v^c \wedge v^d + t_I^a \omega^I , \\ d\omega^I &= \tilde{T}_{aJ}^I v^a \wedge \omega^J . \end{aligned} \quad (3.15)$$

Here, the coefficients t^{ab} , t_I^a and \tilde{T}_{aJ}^I are constants that parameterize the $SU(2)$ structure reduction ansatz for a particular manifold. In particular we exclude any terms on the right-hand side of the above equations involving $SU(2)$ doublets. Note also that in the second equation of (3.15) a possible term proportional to $v^1 \wedge v^2 \wedge v^3$ is immediately set to zero by the constraint that $d(\omega^I \wedge \omega^J \wedge \omega^K) = 0$.

The t^{ab} , t_I^a and \tilde{T}_{aJ}^I specify the torsion classes of Y . We choose them and hence the torsion classes of Y to be constant. These constants are constrained by the fact that the exterior derivative squares to zero and the integral of $d(v^a \wedge v^b \wedge \omega^I \wedge \omega^J)$ over Y should vanish. The constraints are encapsulated by algebraic relations, given by

$$\begin{aligned} t^{ab} \epsilon_{bcd} t^{cd} &= 0 , \\ \epsilon_{bcd} t^{ac} t_I^d + t_J^a \tilde{T}_{bI}^J &= 0 , \\ \tilde{T}_{aJ}^I \eta^{JK} t_K^a &= 0 , \\ \tilde{T}_{bJ}^I t^{ba} - \epsilon^{abc} \tilde{T}_{bK}^I \tilde{T}_{cJ}^K &= 0 , \\ \epsilon_{abc} t^{bc} \eta^{IJ} + \tilde{T}_{aK}^I \eta^{KJ} + \tilde{T}_{aK}^J \eta^{KI} &= 0 . \end{aligned} \quad (3.16)$$

The last equation determines the symmetric part of $\tilde{T}_{jK}^I \eta^{KJ}$, $j = 1, 2$, so that

$$\tilde{T}_{aK}^I \eta^{KJ} = T_{aK}^I \eta^{KJ} - \frac{1}{2} \epsilon_{abc} t^{bc} \eta^{IJ} , \quad (3.17)$$

where T_{aK}^I is a triple of $so(3, n-3)$ matrices, i.e.

$$T_{aK}^I \eta^{KJ} + T_{aK}^J \eta^{KI} = 0 . \quad (3.18)$$

The fourth condition just states that the T_{aJ}^I form an $so(3, n-3)$ subalgebra S defined by

$$\epsilon^{abc} T_{bK}^I T_{cJ}^K = t^{ba} T_{bJ}^I . \quad (3.19)$$

In particular, S is Abelian if $t^{ab} = 0$ and simple if t^{ab} has rank three. The remaining condition is

$$T_{aI}^J t_J^b = \frac{1}{2} \epsilon_{acd} (t^{cd} t_I^b - 2 t^{bc} t_I^d) . \quad (3.20)$$

If t^{ab} is zero, the t_I^a are invariant under S . If t^{ab} is non-zero, the t_I^a form a non-trivial representation under S . The Bianchi identity (3.10) also leads to constraints on the flux numbers appearing in (3.9), given by

$$\begin{aligned} \epsilon_{abc} (\eta^{IJ} g_I^b t_J^c - g_0 t^{bc}) &= 0 , \\ \frac{1}{2} \epsilon_{abc} g_I^a t^{bc} + g_J^a T_{aI}^J &= 0 . \end{aligned} \quad (3.21)$$

3.2 Reduction of gravity

In this section we dimensionally reduce the gravitational term in the eleven-dimensional supergravity action (3.1),

$$S_{\text{grav}} = \frac{1}{2\kappa_{11}^2} \int_{11} (*_{11} 1) r_{11} . \quad (3.22)$$

For this we have to compute the eleven-dimensional Ricci scalar in terms of the ansatz (3.3). We start by computing the connection Ω from the Maurer-Cartan equations (2.20) and (3.15) in Appendix A.2. There we find for the components of the connection

$$\begin{aligned} \tau_\alpha^a &= \frac{1}{2} e^{\rho_3/3} k_b^a t_I^b \omega_{\alpha\beta}^I e^\beta - \frac{1}{4} e^{-\rho_3/3} (k^{-1})_a^b \epsilon_{bcd} t^{cd} e^\alpha + \frac{1}{2} e^{-\rho_3/3} e^{\rho_4/2} (k^{-1})_a^b \zeta_I^{\hat{a}} T_{bJ}^I P_K^J \omega_{\alpha\gamma}^K (I^{\hat{a}})_\beta^\gamma e^\beta , \\ \gamma_\mu^a &= e^{\rho_3/3} k_b^a D_{[\mu} G_{\nu]}^b e^\nu + \frac{1}{2} ((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) K^b + \frac{1}{3} (D_\mu \rho_3) K^a , \\ \phi^a &= -e^{-\rho_3/3} (k_c^a k_d^b t^{cd} - \frac{1}{2} \delta^{ab} k_c^e k_d^e t^{cd}) K^b - \frac{1}{2} \epsilon^{abc} (k^{-1})_b^d D k_d^c , \\ \Psi^{\hat{a}} &= \frac{1}{4} e^{-\rho_3/3} \epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{b}} \zeta^{\hat{c}J} T_{bJ}^I (k^{-1})_a^b K^a - \frac{1}{4} e^{\rho_3/3} e^{-\rho_4/2} \zeta^{\hat{a}I} t_I^b k_b^a K^a - \frac{1}{4} \epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{b}} \zeta^{\hat{c}J} T_{bJ}^I G^b , \\ \lambda_\mu^\alpha &= \frac{1}{4} D_\mu \rho_4 e^\alpha - \frac{1}{2} e^{\rho_4/2} D_\mu \zeta_I^{\hat{a}} \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma e^\beta , \\ \omega_\nu^\mu &= \hat{\omega}_\nu^\mu - e^{\rho_3/3} D_{[\mu} G_{\nu]}^b k_b^a K^a , \end{aligned} \quad (3.23)$$

with the projector $P_J^I = \delta_J^I - \zeta^{\hat{a}I} \zeta_{\hat{a}J}$ and the covariant derivatives given by

$$\begin{aligned} DG^a &= dG^a + \frac{1}{2} t^{ab} \epsilon_{bcd} G^c \wedge G^d , \\ D\rho_3 &= d\rho_3 - \epsilon_{abc} t^{bc} G^a , \\ D\rho_4 &= d\rho_4 + \epsilon_{abc} t^{bc} G^a , \\ Dk_b^a &= dk_b^a - (k_c^a \epsilon_{bde} - \frac{1}{3} k_b^a \epsilon_{cde}) t^{cd} G^e , \\ D\zeta_I^{\hat{a}} &= d\zeta_I^{\hat{a}} - \zeta_K^{\hat{a}} T_{bJ}^K P_I^J G^b . \end{aligned} \quad (3.24)$$

In Appendix A.3 we compute from this connection the components of the ten-dimensional Ricci curvature. For the reduction we only need the Ricci scalar r_{11} , given by

$$\begin{aligned}
r_{11} = & r_4 - 2\nabla^\mu(D_\mu\rho_4) - 2\nabla^\mu(D_\mu\rho_3) \\
& - \frac{5}{4}(D_\mu\rho_4)(D^\mu\rho_4) - \frac{4}{3}(D_\mu\rho_3)(D^\mu\rho_3) - 2D^\mu\rho_4D_\mu\rho_3 \\
& - \frac{1}{2}\text{tr}[k^{-1} \cdot (D_\mu k) \cdot k^{-1} \cdot (D^\mu k)] - \frac{1}{2}\text{tr}[(D_\mu k)^T \cdot (g_3^{-1}) \cdot (D^\mu k)] \\
& + (D_\mu\zeta_I^{\hat{a}})(D^\mu\zeta^{\hat{a}I}) - e^{2\rho_3/3}g_{3ab}D_{[\mu}G_{\nu]}^aD^\mu G^{b\nu} \\
& + \frac{3}{2}e^{-2\rho_3/3}(g_{3ab}t^{ab})^2 - e^{-2\rho_3/3}g_{3ab}g_{3cd}t^{(ac)}t^{(bd)} - \frac{3}{4}e^{-2\rho_3/3}g_3^{ab}\epsilon_{acd}\epsilon_{bef}t^{cd}t^{ef} \\
& - e^{2\rho_3/3}e^{-\rho_4}g_{3ab}t_I^aH^{IJ}t_J^b - \frac{1}{3}e^{-2\rho_3/3}g_3^{ab}\zeta_I^{\hat{a}}\zeta_J^{\hat{a}}T_{aK}^IP^{KL}T_{bL}^J \\
& - 2e^{-\rho_4/2}\epsilon^{\hat{a}\hat{b}\hat{c}}\zeta^{\hat{a}I}\zeta_J^{\hat{b}}\zeta^{\hat{c}K}t_I^aT_{aK}^J,
\end{aligned} \tag{3.25}$$

where r_4 is the four-dimensional Ricci scalar. Note that we can rewrite

$$\begin{aligned}
& - \frac{1}{2}\text{tr}[k^{-1} \cdot (D_\mu k) \cdot k^{-1} \cdot (D^\mu k)] - \frac{1}{2}\text{tr}[(D_\mu k)^T \cdot (g_3^{-1}) \cdot (D^\mu k)] \\
& = - \frac{1}{4}\text{tr}[(D_\mu g_3) \cdot (g_3^{-1}) \cdot (D^\mu g_3) \cdot g_3^{-1}],
\end{aligned} \tag{3.26}$$

with the definition of the covariant derivative as

$$Dg_{3ab} = dg_{3ab} - (g_{3ac}\epsilon_{bde} + g_{3cb}\epsilon_{ade} - \frac{2}{3}g_{3ab}\epsilon_{cde})t^{cd}G^e, \tag{3.27}$$

The eleven-dimensional volume form includes a prefactor $e^{\rho_4+\rho_3}$ that describes the scaling of the internal volume. Thus the reduction of the eleven-dimensional Einstein-Hilbert action to four dimensions in (3.22) is

$$S_{\text{grav}} = \frac{1}{2\kappa_4^2} \int_4 (*_4 1) e^{\rho_4+\rho_3} r_{11}. \tag{3.28}$$

We perform a Weyl rescaling

$$e^\mu \rightarrow e^{-(\rho_3+\rho_4)/2} e^\mu, \tag{3.29}$$

to bring the four-dimensional Einstein-Hilbert term into its canonical form and get

$$\begin{aligned}
S_{\text{grav}} = & \frac{1}{2\kappa_4^2} \int_4 (*_4 1) \left(r_4 - \frac{1}{2}(D_\mu\rho_3)(D^\mu\rho_3) - \frac{3}{4}(D_\mu\phi)(D^\mu\phi) \right. \\
& - \frac{1}{4}\text{tr}[(D_\mu g_3) \cdot (g_3^{-1}) \cdot (D^\mu g_3) \cdot g_3^{-1}] \\
& - H^{IJ}(D_\mu\zeta_I^{\hat{a}})(D^\mu\zeta_J^{\hat{a}}) - e^{\phi+\rho_3}g_{3ab}D_{[\mu}G_{\nu]}^aD^\mu G^{b\nu} \\
& + \frac{3}{2}e^{-\phi-\rho_3}(g_{3ab}t^{ab})^2 - e^{-\phi-\rho_3}g_{3ab}g_{3cd}t^{(ac)}t^{(bd)} \\
& - \frac{3}{4}e^{-\phi-\rho_3}g_3^{ab}\epsilon_{acd}\epsilon_{bef}t^{cd}t^{ef} - e^{-2\phi+\rho_3}g_{3ab}t_I^aH^{IJ}t_J^b \\
& \left. - \frac{1}{3}e^{-\phi-\rho_3}g_3^{ab}\zeta_I^{\hat{a}}\zeta_J^{\hat{a}}T_{aK}^IP^{KL}T_{bL}^J - 2e^{-\frac{3}{2}\phi}\epsilon^{\hat{a}\hat{b}\hat{c}}\zeta^{\hat{a}I}\zeta_J^{\hat{b}}\zeta^{\hat{c}K}t_I^aT_{aK}^J \right),
\end{aligned} \tag{3.30}$$

where we defined $\phi = \rho_4 + \frac{2}{3}\rho_3$ with

$$D\phi = d\phi + \frac{1}{3}\epsilon_{abc}t^{bc}G^a. \tag{3.31}$$

3.3 Reduction of the four-form field strength

Next we want to reduce the four-form field strength action

$$S_{G_4} = -\frac{1}{4\kappa_{11}^2} \int_{11} G_4 \wedge *_{11} G_4 - \frac{1}{12\kappa_{11}^2} \int_{11} G_4 \wedge G_4 \wedge C_3 . \quad (3.32)$$

For this we compute the four-form field strength G_4 , defined in (3.8), using (3.15) and (3.9). We find

$$\begin{aligned} G_4 = & (e^{-3\phi-\rho_3} f_0 (*_4 1) + d\hat{C} + \hat{C}_a \wedge DG^a) + (D\hat{C}_a - \epsilon_{abc} C^b \wedge DG^c) \wedge (v^a + G^a) \\ & + (DC_I + c_{aI} DG^a) \wedge \omega^I + \frac{1}{2} \epsilon_{abc} (DC^a + c_0 DG^a) \wedge (v^b + G^b) \wedge (v^c + G^c) \\ & + \frac{1}{6} DC_0 \wedge \epsilon_{abc} (v^a + G^a) \wedge (v^b + G^b) \wedge (v^c + G^c) + Dc_{aI} \wedge (v^a + G^a) \wedge \omega^I \\ & + (g_I^a + c_0 t_I^a + \epsilon^{abc} T_{bI}^J c_{cJ} + t^{(ab)} c_{bI}) \frac{1}{2} \epsilon_{ade} (v^d + G^d) \wedge (v^e + G^e) \wedge \omega^I \\ & + (g_0 + c_{aI} t_J^a \eta^{IJ}) \text{vol}_4^{(0)} , \end{aligned} \quad (3.33)$$

where we defined DG^a in (3.23) and the other covariant derivatives are

$$\begin{aligned} D\hat{C}_a = & d\hat{C}_a - \epsilon_{abc} t^{db} G^c \wedge \hat{C}_d , \\ DC_I = & dC_I + t_I^a \hat{C}_a - T_{aI}^J G^a \wedge C_J + \frac{1}{2} \epsilon_{abc} t^{bc} G^a \wedge C_I + \frac{1}{2} g_I^a \epsilon_{abc} G^b \wedge G^c , \\ DC^a = & dC^a + t^{ba} \hat{C}_b + t^{ab} \epsilon_{bcd} G^c \wedge C^d - \epsilon_{bcd} t^{cd} G^b \wedge C^a , \\ DC_0 = & dC_0 - \epsilon_{abc} t^{bc} (C^a + c_0 G^a) , \\ Dc_{aI} = & dc_{aI} - T_{aI}^J C_J + \frac{1}{2} \epsilon_{abc} t^{bc} C_I + \epsilon_{abc} t_I^b C^c - \epsilon_{abc} (g_I^b + t^{db} c_{dI}) G^c . \end{aligned} \quad (3.34)$$

Now we can insert this into (3.32) and perform the Weyl rescaling (3.29). We find for the kinetic term

$$\begin{aligned} S_{\text{kin}} = & -\frac{1}{4\kappa_4^2} \int_4 (e^{3\phi+\rho_3} (d\hat{C} + \hat{C}_a \wedge DG^a) \wedge *_4 (d\hat{C} + \hat{C}_b \wedge DG^b) \\ & - e^{2\phi} g_3^{ab} (D\hat{C}_a + \epsilon_{acd} C^c \wedge DG^d) \wedge *_4 (D\hat{C}_b + \epsilon_{bef} C^e \wedge DG^f) \\ & + e^{\rho_3} H^{IJ} (DC_I + c_{aI} DG^a) \wedge *_4 (DC_J + c_{bJ} DG^b) \\ & + e^{\phi-\rho_3} g_{3ab} (DC^a + c_0 DG^a) \wedge *_4 (DC^b + c_0 DG^b) \\ & - e^{-2\rho_3} DC_0 \wedge *_4 DC_0 - e^{-\phi} H^{IJ} g_3^{ab} Dc_{aI} \wedge *_4 Dc_{bJ} \\ & + e^{-2\phi} (g_I^a + c_0 t_I^a + \epsilon^{acd} T_{cI}^K c_{dK} + t^{(ac)} c_{cI}) H^{IJ} g_{3ab} \\ & \quad (g_J^b + c_0 t_J^b + \epsilon^{bef} T_{eJ}^L c_{fL} + t^{(be)} c_{eI}) \\ & + e^{-3\phi+\rho_3} |g_0 + c_{aI} t_J^a \eta^{IJ}|^2) . \end{aligned} \quad (3.35)$$

To evaluate the topological term correctly in the presence of four-form flux, we assume eleven-dimensional spacetime to be the boundary of a fictional twelve-dimensional space

and write the topological term of (3.32) as [21, 22]

$$\begin{aligned}
S_{\text{top}} &= -\frac{1}{12\kappa_{11}^2} \int_{12} G_4 \wedge G_4 \wedge G_4 \\
&= -\frac{1}{4\kappa_{11}^2} \int_{11} G_4^{\text{flux}} \wedge dC^T \wedge C^T - \frac{1}{4\kappa_{11}^2} \int_{11} G_4 \wedge G_4 \wedge C^V \\
&\quad + \frac{1}{4\kappa_{11}^2} \int_{11} G_4 \wedge dC^V \wedge C^V - \frac{1}{12\kappa_{11}^2} \int_{11} dC^V \wedge dC^V \wedge C^V ,
\end{aligned} \tag{3.36}$$

where we used that the flux G_4^{flux} squares to zero, cf. (3.9), and defined C^T to be the part of C_3 with two or more external legs, while C^V is the component of C_3 with one or less external leg. In other words,

$$\begin{aligned}
C^V &= C_I \wedge \omega^I + \frac{1}{2} \epsilon_{abc} C^a \wedge (v^b + G^b) \wedge (v^c + G^c) + c_{aI} (v^a + G^a) \wedge \omega^I \\
&\quad + \frac{1}{6} c_0 \epsilon_{abc} (v^a + G^a) \wedge (v^b + G^b) \wedge (v^c + G^c) ,
\end{aligned} \tag{3.37}$$

and

$$-\frac{1}{4\kappa_{11}^2} \int_{11} G_4^{\text{flux}} \wedge dC^T \wedge C^T = -\frac{1}{4\kappa_4^2} \int_4 (g_0 t^{ab} + g_I^a \eta^{IJ} t_J^b) \hat{C}_a \wedge \hat{C}_b . \tag{3.38}$$

Now we integrate out the three-form \hat{C} , or, more easily, integrate out its field strength $\hat{F} = d\hat{C} + \hat{C}_a \wedge DG^a$, since \hat{C} does not appear by itself in the action. The equation of motion for \hat{F} is

$$\hat{F} = -e^{-3\phi-\rho_3} (f_0 + (g_0 + c_{aI} t_J^a \eta^{IJ}) c_0 + (g_I^a + \frac{1}{2} \epsilon^{abc} T_{bI}^J c_{cJ} + \frac{1}{2} t^{(ab)} c_{bI}) \eta^{IK} c_{aK}) (*_4 1) , \tag{3.39}$$

and inserting this for \hat{F} in the action leads to an additional term for the potential

$$S_{\hat{F}} = \frac{1}{4\kappa_4^2} \int_4 (*_4 1) e^{-3\phi-\rho_3} |(g_0 + c_{aI} t_J^a \eta^{IJ}) c_0 + (g_I^a + \frac{1}{2} \epsilon^{abc} T_{bI}^J c_{cJ} + \frac{1}{2} t^{(ab)} c_{bI}) \eta^{IJ} c_{aJ}|^2 . \tag{3.40}$$

Note that the topological term then reduces to

$$\begin{aligned}
S_{\text{top}} &= -\frac{1}{4\kappa_{11}^2} \int_4 (c_0 \eta^{IJ} (DC_I + c_{aI} DG^a) \wedge (DC_J + c_{bJ} DG^b) \\
&\quad + (D\hat{C}_a - \epsilon_{ade} C^d \wedge DG^e) \\
&\quad \wedge (\epsilon^{abc} c_{cI} \eta^{IJ} DC_{bJ} + 2f_0 G^a + 2g_0 C^a + 2g_I^a \eta^{IJ} C_J) \\
&\quad + c_{aI} \eta^{IJ} (2DC_J + c_{bJ} DG^b) \wedge DC^a) + (g_0 t^{ab} + g_I^a \eta^{IJ} t_J^b) \hat{C}_a \wedge \hat{C}_b \\
&\quad + \eta^{IJ} (DC_I - t_I^a \hat{C}_a) \wedge C^b \wedge (\frac{1}{2} \epsilon_{bcd} t_J^c C^d - T_{bJ}^K C_K + \frac{1}{2} \epsilon_{bcd} t^{cd} C_J) \\
&\quad - (f_0 G^a + \frac{1}{2} g_0 C^a) \wedge \epsilon_{abc} C^b \wedge DG^c ,
\end{aligned} \tag{3.41}$$

where $S_{\text{top,vec}}$ only depends on the vector fields C_I and C^a .

Finally, we want to introduce scalar fields γ^a so that the kinetic term of the \hat{C}_a can be replaced. To be consistent, we also have to introduce magnetic vector fields \tilde{C}_I and \tilde{C}_a that are dual to C_I and C^a , as well as a number of auxiliary two-form fields. Also,

we want to perform an electric-magnetic duality between C^a and \tilde{C}_a to end up in the standard frame of $N = 4$ gauged supergravity. Since the scalars c_{aI} and c_0 are charged under C^a , \tilde{C}_I and \tilde{C}_a will be charged under their dual two-forms \hat{C}_I^a and \hat{C}_0 , which must be introduced as well. Note that this very much complicates the situation compared to [1, 18].

When dualizing fields, Bianchi identities and field equations are swapped. This means that from the Bianchi identities of $\hat{F}_a = D\hat{C}_a - \epsilon_{abc}C^b \wedge DG^c$ and $F^a = DC^a + c_0 DG^a$ we can deduce the couplings of their dual fields γ^a and \tilde{C}_a in the Lagrangian. On the other hand, the field equations of \hat{F}_a and F^a tell us what should be the covariant derivatives of γ^a and \tilde{C}_a . In particular, we can see that if C^a appears in the covariant derivative of scalar fields, their dual tensors have to appear in the covariant derivative of \tilde{C}_a , and there should be an additional topological coupling of this tensor to C^a in the final Lagrangian. Furthermore, if \hat{C}_a appears in the covariant derivative of an electric gauge field, the scalar γ^a must be gauged under the magnetic dual of the gauge field, and \hat{C}_a should be topologically coupled to this magnetic vector.

In Appendix B.1 we perform the duality transformation from \hat{C}_a and C^a to γ^a and \tilde{C}_a . Both \hat{C}_a and C^a become auxiliary fields without kinetic terms. Moreover, also a new auxiliary vector field \tilde{C}_I and the auxiliary tensors \hat{C}_0 and \hat{C}_I^a appear in the dual Lagrangian. These dual fields will mostly appear through their covariant derivatives

$$\begin{aligned}
D\hat{C}_0 &= d\hat{C}_0 + \epsilon_{abc}t^{bc}G^a \wedge C_0 - \epsilon_{abc}t_I^a\eta^{IJ}G^b \wedge \hat{C}_J^c - \frac{1}{2}t_I^a\eta^{IJ}C_J \wedge \hat{C}_a \\
&\quad - f_0\epsilon_{abc}G^a \wedge G^b \wedge G^c, \\
D\hat{C}_I^a &= d\hat{C}_I^a - T_{bI}^J G^a \wedge \hat{C}_J^b - \frac{1}{2}\epsilon_{bcd}t^{cd}G^a \wedge \hat{C}_I^b - t^{ba}C_I \wedge \hat{C}_b, \\
D\tilde{C}_a &= d\tilde{C}_a - \epsilon_{abc}t^{bc}\hat{C}_0 - \epsilon_{abc}t_I^b\eta^{IJ}\hat{C}_J^c - g_0\hat{C}_a - \frac{1}{2}T_{aK}^I\eta^{KJ}C_I \wedge C_J \\
&\quad + \epsilon_{bcd}t^{bc}G^d \wedge \tilde{C}_a - \epsilon_{abc}t^{db}G^c \wedge \tilde{C}_d + \frac{1}{2}f_0\epsilon_{abc}G^b \wedge G^c, \\
D\tilde{C}_I &= d\tilde{C}_I + T_{aI}^J\hat{C}_J^a + \frac{1}{2}\epsilon_{abc}t^{bc}\hat{C}_I^a - g_I^a\hat{C}_a + \frac{1}{2}t_I^a\epsilon_{abc}C^b \wedge C^c - g_I^a\epsilon_{abc}G^b \wedge C^c \\
&\quad + T_{aI}^J C^a \wedge C_J + \frac{1}{2}\epsilon_{abc}t^{bc}C^a \wedge C_I - T_{aI}^J G^a \wedge \tilde{C}_J - \frac{1}{2}\epsilon_{abc}t^{bc}G^a \wedge \tilde{C}_I, \\
D\gamma^a &= d\gamma^a - (f_0\delta_d^a + t^{ab}\epsilon_{bcd}(\gamma^c + \frac{1}{2}\epsilon^{cef}c_{eI}\eta^{IJ}c_{fJ}))G^d - t^{ab}\tilde{C}_b - t_I^a\eta^{IJ}\tilde{C}_J - g_0C^a \\
&\quad - g_I^a\eta^{IJ}C_J + c_{bI}\eta^{IJ}(t_J^{[a}C^{b]} - g_J^{[a}G^{b]}) + \frac{1}{2}\epsilon^{abc}c_{bI}\eta^{IJ}T_{cJ}^K C_K + \frac{1}{2}t^{[ab]}c_{bI}\eta^{IJ}C_J.
\end{aligned} \tag{3.42}$$

The kinetic terms in the dual Lagrangian can be computed to

$$\begin{aligned}
\tilde{\mathcal{L}}_{\text{kin}} &= -\frac{1}{4\kappa_4^2} \int_4 \left(e^{\rho_3} H^{IJ} (DC_I + c_{aI} DG^a) \wedge *_4 (DC_J + c_{bJ} DG^b) \right. \\
&\quad + e^{\rho_3 - \phi} g_3^{ab} (D\tilde{C}_a - c_{aI}\eta^{IK} DC_K - \epsilon_{acd}\gamma^c DG^d - \frac{1}{2}c_{aI}\eta^{IK} c_{cK} DG^c) \\
&\quad \wedge * (D\tilde{C}_b - c_{bJ}\eta^{JL} DC_L - \epsilon_{bef}\gamma^e DG^f - \frac{1}{2}c_{bJ}\eta^{JL} c_{eL} DG^e) \\
&\quad - e^{-2\rho_3} DC_0 \wedge *_4 DC_0 - e^{-\phi} H^{IJ} g_3^{ab} Dc_{aI} \wedge *_4 Dc_{bJ} \\
&\quad \left. - e^{-2\phi} g_{3ab} (D\gamma^a + \frac{1}{2}\epsilon^{acd}c_{cI}\eta^{IJ} Dc_{dJ}) \wedge *_4 (D\gamma^b + \frac{1}{2}\epsilon^{bef}c_{eI}\eta^{IJ} Dc_{fJ}) \right),
\end{aligned} \tag{3.43}$$

the potential is given by

$$\begin{aligned}
\tilde{S}_{\text{pot}} = & -\frac{1}{4\kappa_4^2} \int_4 \left(e^{-2\phi} (g_I^a + c_0 t_I^a + \epsilon^{acd} T_{cI}^K c_{dK} + t^{(ac)} c_{cI}) H^{IJ} g_{3ab} \right. \\
& (g_J^b + c_0 t_J^b + \epsilon^{bef} T_{eJ}^L c_{fL} + t^{(be)} c_{eJ}) \\
& + e^{-3\phi-\rho_3} |f_0 + (g_0 + c_{aI} t_J^a \eta^{IJ}) c_0 + (g_I^a + \frac{1}{2} \epsilon^{abc} T_{bI}^J c_{cJ} + \frac{1}{2} t^{(ab)} c_{bI}) \eta^{IJ} c_{aJ}|^2 \\
& \left. + e^{-3\phi+\rho_3} |g_0 + c_{aI} t_J^a \eta^{IJ}|^2 \right), \tag{3.44}
\end{aligned}$$

and the topological terms are

$$\begin{aligned}
\tilde{S}_{\text{top}} = & -\frac{1}{4\kappa_4^2} \int_4 \left(c_0 (\eta^{IJ} DC_I \wedge DC_J - 2D\tilde{C}_a \wedge DG^a) \right. \\
& + g_0 (2DC^a - t^{ba} \hat{C}_b) \wedge \hat{C}_a + 2\epsilon_{abc} t_I^a \eta^{IJ} \tilde{C}_J \wedge C^b \wedge DG^c \\
& + 2DC^a \wedge \epsilon_{abc} (t^{bc} \hat{C}_0 + t_I^b \eta^{IJ} \hat{C}_J^c) - t_I^a \eta^{IJ} (2D\tilde{C}_J - g_J^b \hat{C}_b) \wedge \hat{C}_a \\
& + \epsilon_{abc} t^{bc} \eta^{IJ} C^a \wedge C_I \wedge DC_J + t_I^a \eta^{IJ} \epsilon_{abc} C^a \wedge C^b \wedge DC_J \\
& \left. + \frac{1}{6} f_0 \epsilon_{def} t^{ef} \epsilon_{abc} G^a \wedge G^b \wedge G^c \wedge C^d \right). \tag{3.45}
\end{aligned}$$

Variation with respect to the auxiliary tensor fields leads to the duality relations between electric and magnetic vectors

$$\begin{aligned}
DC_a + c_0 DG^a &= e^{\rho_3-\phi} g_3^{ab} * (D\tilde{C}_b - c_{bI} \eta^{IK} DC_K - \epsilon_{bcd} \gamma^c DG^d - \frac{1}{2} c_{bI} \eta^{IK} c_{cK} DG^c), \\
D\tilde{C}_I - c_{aI} DC^a &= e^{\rho_3} H_I^J * (DC_J + c_{aI} DG^a) + c_0 (DC_I + c_{aI} DG^a), \tag{3.46}
\end{aligned}$$

while variation with respect to the magnetic vectors gives the duality relations between tensors and scalars

$$\begin{aligned}
D\hat{C}_a - \epsilon_{abc} C^b \wedge DG^c &= e^{-2\phi} g_{3ab} * (D\gamma^b + \frac{1}{2} \epsilon^{acd} c_{cI} \eta^{IJ} Dc_{dJ}), \\
D\hat{C}_0 + \frac{1}{2} \eta^{IJ} C_I \wedge F_J &= e^{-2\rho_3} * Dc_0, \\
D\hat{C}_I^a - C^a \wedge DC_I - c_{bI} C^b \wedge DG^a - \epsilon^{abc} c_{bI} D\hat{C}_c &= e^{-\phi} H_I^J g_3^{ab} * Dc_{bJ}. \tag{3.47}
\end{aligned}$$

Note that each of the relations in (3.47) and (3.46) gets multiplied by certain charge components. Thus, if certain charges are vanishing, the corresponding duality equation is eliminated. At the same time, the corresponding couplings in the covariant derivative vanish and the corresponding auxiliary field is removed from the Lagrangian altogether. In the generic case of non-vanishing couplings, we can use the duality relations (3.47) and (3.46) in order to eliminate the fields we have introduced above and come back to the Lagrangian of (3.35) and (3.41) that we obtained from the reduction.

The action obtained in (3.43), (3.44) and (3.45) together with (3.30) fits perfectly into the framework of $N = 4$ gauged supergravity. We make the identification with the standard notation in the next section.

4 Matching with $N = 4$ supergravity

In this section we want to match the results of the dimensional reduction on a seven-dimensional $SU(2)$ -structure manifold with the standard formulation of $N = 4$ gauged supergravity, which is reviewed in Appendix C. We organize the vector fields as

$$V^{M+} = (G^a, \tilde{C}_a, \eta^{IJ} C_J) , \quad V^{M-} = (C^a, \tilde{G}_a, -\eta^{IJ} \tilde{C}_J) , \quad (4.1)$$

so that the metric η_{MN} is in the standard form

$$\eta_{MN} = \begin{pmatrix} 0 & -\delta_a^b & 0 \\ -\delta_b^a & 0 & 0 \\ 0 & 0 & \eta_{IJ} \end{pmatrix} . \quad (4.2)$$

Note that this involves an electric-magnetic duality transformation between \tilde{C}_a and C^a , which can be performed in the standard way following [8].

Next, the scalars c_0 and ρ_3 combine into the $N = 4$ axiodilaton τ as $\tau = (-c_0 + i e^{\rho_3})$ so that the covariant derivative reads

$$D\tau = d\tau - \epsilon_{abc} t^{bc} (C^a + \tau G^a) . \quad (4.3)$$

Therefore we find for the embedding tensor components $\xi_{\alpha M}$ that the only non-vanishing component is

$$\xi_{+a} = -\epsilon_{abc} t^{bc} . \quad (4.4)$$

The coset matrix M_{MN} of the Grassmanian

$$\frac{SO(6, n)}{SO(6) \times SO(n)} \quad (4.5)$$

is given in terms of the scalars $(\phi, g_{3ab}, \zeta_I^{\hat{a}}, \gamma^a, c_{aI})$ by

$$\begin{aligned} M_{ab} &= e^\phi g_{3ab} - \frac{1}{4} e^{-\phi} g_3^{cd} (\epsilon_{ace} \gamma^e + c_{aI} \eta^{IJ} c_{cJ}) (\epsilon_{bdf} \gamma^f + c_{bK} \eta^{KL} c_{dL}) + H^{IJ} c_{aI} c_{bJ} , \\ M_b^a &= \frac{1}{2} e^{-\phi} g_3^{ac} (\epsilon_{cbd} \gamma^d + c_{cI} \eta^{IJ} c_{bJ}) , \\ M^{ab} &= e^{-\phi} g_3^{ab} , \\ M_{aI} &= H_I^J c_{aJ} + \frac{1}{2} e^{-\phi} (\epsilon_{abc} \gamma^b + c_{aI} \eta^{IJ} c_{bJ}) g_3^{cd} c_{dI} , \\ M_I^a &= e^{-\phi} g_3^{ab} c_{bI} , \\ M_{IJ} &= H_{IJ} + e^{-\phi} g_3^{ab} c_{aI} c_{bJ} . \end{aligned} \quad (4.6)$$

The corresponding vielbein \mathcal{V} is given by

$$\begin{aligned}
\mathcal{V}^b_a &= e^{\phi/2} k_a^b + e^{-\phi/2} (k^{-1})^c_b (\epsilon_{cad} \gamma^d + c_{cI} \eta^{IJ} c_{aJ}) , \\
\mathcal{V}^{ba} &= e^{-\phi/2} (k^{-1})^a_b , \\
\mathcal{V}^b_I &= e^{-\phi/2} (k^{-1})^c_b c_{cI} , \\
\mathcal{V}^{\hat{b}}_a &= \zeta^{\hat{b}I} c_{aI} , \\
\mathcal{V}^{\hat{b}a} &= 0 , \\
\mathcal{V}^{\hat{b}}_I &= \zeta^{\hat{b}}_I ,
\end{aligned} \tag{4.7}$$

so that

$$M_{MN} = 2\mathcal{V}_M^a \mathcal{V}_N^a + 2\mathcal{V}_M^{\hat{a}} \mathcal{V}_N^{\hat{a}} - \eta_{MN} . \tag{4.8}$$

From the covariant derivatives we can read off the remaining embedding tensor components

$$\begin{aligned}
f_{+abc} &= f_0 \epsilon_{abc} , \\
f_{+ab}{}^c &= -\epsilon_{abd} t^{cd} , \\
f_{+aIJ} &= \eta_{IK} T_{aJ}^K , \\
f_{+abI} &= \epsilon_{abc} g_I^c , \\
f_{-abc} &= g_0 \epsilon_{abc} , \\
f_{-abI} &= \epsilon_{abc} t_I^c .
\end{aligned} \tag{4.9}$$

Moreover, we identify

$$\begin{aligned}
B^{++} &= 2\hat{C}_0 , \\
B^{ab} &= \epsilon^{abc} \hat{C}_c + C^a \wedge G^b , \\
B^{aI} &= \eta^{IJ} (2\hat{C}_J^a + C^a \wedge C_J - G^a \wedge \tilde{C}_J) .
\end{aligned} \tag{4.10}$$

Comparing the charges (4.4) and (4.9) with [8], we see that the remaining auxiliary two-forms as well as the magnetic vector \tilde{G}_a do not explicitly appear in the Lagrangian. Moreover one can check that the constraints (3.16), (3.20) and (3.21) solve the quadratic constraints (C.9).

5 Consistent truncation

In this section we want to show that the $SU(2)$ reduction to four dimensions is indeed a consistent truncation of the eleven-dimensional supergravity action. In other words, we want to show that the four-dimensional equations of motion imply the ten-dimensional ones. In [1] a heuristic argument was already given why $SU(2)$ structure reductions for modes ω^I and v^a that obey the constraints (3.2) and (3.15) are consistent truncations.

In this section we will prove this claim by an explicit check of the eleven-dimensional equations of motion.

The four-dimensional relevant equations are the Einstein equation

$$\begin{aligned}\hat{R}_{\mu\nu} = & -2\text{tr}(((D_\mu \mathcal{V})\mathcal{V}^{-1} + (\mathcal{V}^{-1})^T(D\mathcal{V})^T)^2((D_\nu \mathcal{V})\mathcal{V}^{-1} + (\mathcal{V}^{-1})^T(D\mathcal{V})^T)) \\ & - \frac{1}{2}(\text{Im } \tau)^{-2}D_{(\mu}\tau D_{\nu)}\bar{\tau} - \text{Im}(\tau)M_{MN}DV_{\mu\rho}^{M+}DV_{\nu}^{N+\rho} + \frac{1}{2}g_{\mu\nu}(\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}}) ,\end{aligned}\quad (5.1)$$

the equations of motion for the vector fields

$$\begin{aligned}D(\text{Re}(\tau)DV^{M+} + \text{Im}(\tau)M^{MP}\eta_{PN}DV^{N+}) = & -\eta^{MN}(\xi_{\beta M}M_{+\gamma} *_4 DM^{\beta\gamma} \\ & + \frac{1}{2}\theta_{MP}^N M_{NQ} *_4 DM^{QP}) ,\end{aligned}\quad (5.2)$$

and for the scalars

$$\begin{aligned}& D((\mathcal{V}^{-1})_M^n *_4 ((D\mathcal{V})\mathcal{V}^{-1} + (\mathcal{V}^{-1})^T(D\mathcal{V})^T)_{mn}) - \frac{1}{4}\text{Im}(\tau)\mathcal{V}_N^m DV^{M+} \wedge *_4 DV^{N+} \\ = & \frac{1}{16}(M^{\alpha\beta}f_{\alpha MPR}f_{\beta NQS}(\mathcal{V}^{-1})_m^N(M^{PQ}M^{RS} - \eta^{PQ}\eta^{RS}) + 3M^{\alpha\beta}\mathcal{V}_N^m \xi_\alpha^M \xi_\beta^N) \\ & - \frac{2}{3}\epsilon^{\alpha\beta}f_{\alpha\tilde{M}NP}f_{\beta QRS}\mathcal{V}_M^m M^{MNPQRS} , \\ & D^\mu((\text{Im } \tau)^{-2}D_\mu\tau) - \frac{1}{4}(\eta_{MN} + i M_{MN} *_4)DV^{M+} \wedge DV^{N+} \\ = & \frac{1}{8}(\partial_\tau M^{\alpha\beta})(f_{\alpha MNP}f_{\beta QRS}(\frac{1}{3}M^{MQ}M^{NR}M^{PS} + (\frac{2}{3}\eta^{MQ} - M^{MQ})\eta^{NR}\eta^{PS}) \\ & + 3\xi_\alpha^M \xi_\beta^N M_{MN}) ,\end{aligned}\quad (5.3)$$

as well as the identities for the auxiliary field strengths³

$$\begin{aligned}DV^{M-} = & -\text{Re}(\tau)DV^{M+} - \text{Im}(\tau)M^{MP}\eta_{PN} *_4 DV^{N+} , \\ \eta_{MN} *_4 DDV^{N+} = & -\xi_{\alpha M}M_{-\beta}DM^{\alpha\beta} - \frac{1}{2}\Theta_{MP}^N M_{NQ}DM^{QP} .\end{aligned}\quad (5.4)$$

originating from the Lagrangian of [8] discussed in Appendix C. The eleven-dimensional equations of motion originate from the action (3.1) and are given by

$$\begin{aligned}\text{Ric}_{AB} - \frac{1}{2}g_{AB}r_{11} = & \frac{1}{12}(G_{ACDE}G_B^{CDE} - \frac{1}{4}g_{AB}G_{CDEF}G^{CDEF}) , \\ d *_4 G_4 = & -\frac{1}{2}G_4 \wedge G_4 .\end{aligned}\quad (5.5)$$

The major work consists of showing that the eleven-dimensional Einstein equations are satisfied if the four-dimensional equations of motion (and Bianchi identities) hold. The technical details for determining the Ricci curvature and the energy-momentum tensor are delegated to the first two appendices. In Appendix A.4 we give the Ricci curvature in the Einstein frame. In Appendix B.2, we also compute the energy-momentum tensor generated by G_4 . When we insert these results into the eleven-dimensional Einstein equation, we see that the equations reduce to the four-dimensional equations of motion in the following way:

- The trace of the Einstein equations is satisfied by the equations of motion for ϕ , ρ_3 and by the trace of the four-dimensional Einstein equation.

³The covariant derivative of DV^+ gives via the Bianchi identity the three-form field strengths of the auxiliary tensor fields, cf. [8].

- The Einstein equations with indices $(\mu\nu)$ give the four-dimensional Einstein equations.
- For the indices (μa) we recover the equations of motion for the Kaluza-Klein vector G_{μ}^a .
- The trace of the (ab) component of the eleven-dimensional Einstein equations gives the equation of motion for ρ_3 , while the traceless part is the equation of motion for k_b^a .
- The trace of the $(\alpha\beta)$ component of the eleven-dimensional Einstein equations is the equation of motion for ρ_4 , i.e. for $\phi - 2\rho_3/3$. The traceless part is the equation of motion for $\zeta_I^{\hat{a}}$.

For the higher form field components we then use the four-dimensional equation of motion for \hat{F} (3.39) to eliminate \hat{F} , the scalar-tensor duality relation (3.47) to replace the tensor fields by their dual scalars and the electro-magnetic duality relation (3.46) to replace all magnetic vector fields by their electric counterparts. In this way we can rewrite the eleven-dimensional Einstein equations in terms of four-dimensional scalars and electric vector fields (up to appearances of the magnetic vectors and tensors in the gaugings). As expected, this completely reproduces the four-dimensional equations of motions for these scalars and vector fields.

6 Simple supersymmetric backgrounds

Let us now briefly discuss some classes of supersymmetric AdS vacua. We will only discuss the simple examples of $N = 4$ AdS vacua discussed in [23] and of $N = 3$ AdS vacua from Tri-Sasakian manifolds whose consistent truncation has been worked out in [17]. The discussion of cases with $N \leq 2$ goes beyond the scope of this paper.

6.1 $N = 4$ AdS vacua

In [23] four-dimensional $N = 4$ AdS vacua had been classified. A necessary requirement for such backgrounds is that there is one electrically and one magnetically gauged $SU(2)$ in the theory whose gauge bosons are graviphotons. Let us apply the findings of [23] to the gaugings of $SU(2)$ structures given in (4.4) and (4.9). The embedding tensor component ξ in (4.4) must be zero in these vacua, which means that t^{ab} is symmetric. Moreover, the electric and magnetic gaugings obey the relationship

$$f_+ = \text{Re } \tau f_- + \text{Im } \tau *_6 f_- , \quad (6.1)$$

where $*_6$ is the Hodge star in the six-dimensional space of graviphotons and τ is the axiodilaton. For the possible gaugings given in (4.9), this means that

$$\mathcal{V}^{ad} \mathcal{V}^{\hat{b}I} \mathcal{V}^{\hat{c}J} f_{+dIJ} \neq 0 , \quad (6.2)$$

where the supergravity vielbein has been given in (4.7). But $N = 4$ supersymmetry also requires that

$$\tilde{\mathcal{V}}^{\tilde{a}d} \mathcal{V}^{\hat{b}I} \mathcal{V}^{\hat{c}J} f_{+dIJ} = 0 , \quad (6.3)$$

where the dual vielbein $\tilde{\mathcal{V}}^{\tilde{a}}$ is given by

$$\begin{aligned} \tilde{\mathcal{V}}^{\tilde{b}}_a &= -e^{\phi/2} k_a^{\tilde{b}} - e^{-\phi/2} (k^{-1})^c_{\tilde{b}} (\epsilon_{cad} \gamma^d + c_{cI} \eta^{IJ} c_{aJ}) , \\ \tilde{\mathcal{V}}^{\tilde{b}a} &= e^{-\phi/2} (k^{-1})^a_{\tilde{b}} , \\ \tilde{\mathcal{V}}^{\tilde{b}}_I &= e^{-\phi/2} (k^{-1})^c_{\tilde{b}} c_{cI} . \end{aligned} \quad (6.4)$$

Comparing the two formulas (6.2) and (6.3), using (4.7) and the above formula, shows the contradiction. Thus, in the class of supergravities obtained from $SU(2)$ structure truncations no $N = 4$ AdS vacua can be found.

6.2 Tri-Sasakian manifolds

The consistent truncations worked out in [17] for Tri-Sasakian manifolds admit $N = 3$ AdS vacua. These truncations are minimal in that I runs $I = 1, 2, 3$. There the ansatz is

$$g_0 = 0 , \quad g_I^a = 0 , \quad t^{ab} = -2\delta^{ab} , \quad t_I^a = 2\delta_I^a , \quad T_{aJ}^I = -2\epsilon_{aIJ} . \quad (6.5)$$

A classification of $N = 3$ vacua is beyond the scope of this paper, but note that one could easily for instance add a non-trivial four-form flux by switching on $g_I^a \sim \delta_I^a$. These four-form fluxes effectively just change the value of the axiodilaton τ but do not modify the solution in any other way.

An interesting special case is the reduction on S^7 . We have already shown that the discussed $SU(2)$ structure reductions do not allow for $N = 4$ AdS vacua. And indeed, the consistent truncation presented here does not distinguish between S^7 and any other Tri-Sasakian manifold. Thus while the full vacuum of $AdS_4 \times S^7$ preserves $N = 8$ supersymmetry, the truncation to this $N = 4$ gauged supergravity (i.e. the truncation to $SU(2)$ singlet and triplet modes) preserves only $N = 3$.

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A Curvature computations

A.1 Useful formulas

From (3.4) we find the identities

$$(k^{-1})^b_a = \frac{1}{2} \epsilon_{acd} \epsilon^{bef} k_e^c k_f^d . \quad (A.1)$$

It also implies

$$(k^{-1})_a^b dk_b^a = 0 , \quad k_d^a \epsilon_{abc} = \epsilon_{def} (k^{-1})_b^e (k^{-1})_c^f . \quad (\text{A.2})$$

From the algebra (3.15) we find

$$d(\epsilon_{abc} v^b \wedge v^c) = 2\epsilon_{abc} t^{bc} \text{vol}_3^{(0)} + 2\epsilon_{abc} t_I^b v^c \wedge \omega^I , \quad (\text{A.3})$$

and

$$d\text{vol}_4^{(0)} = -\epsilon_{abc} t^{bc} v^a \wedge \text{vol}_4^{(0)} . \quad (\text{A.4})$$

Another useful formula will be

$$d(v^a + G^a) = DG^a - t^{ab} \epsilon_{bcd} G^c \wedge (v^d + G^d) + \frac{1}{2} t^{ab} \epsilon_{bcd} (v^c + G^c) \wedge (v^d + G^d) + t_I^a \omega^I , \quad (\text{A.5})$$

where we defined the covariant derivative

$$DG^a = dG^a + \frac{1}{2} t^{ab} \epsilon_{bcd} G^c \wedge G^d , \quad (\text{A.6})$$

and

$$d\omega^I = -T_{aJ}^I G^a \wedge \omega^J + \frac{1}{2} \epsilon_{abc} t^{bc} G^a \wedge \omega^I + T_{aJ}^I (v^a + G^a) \wedge \omega^J - \frac{1}{2} \epsilon_{abc} t^{bc} (v^a + G^a) \wedge \omega^I . \quad (\text{A.7})$$

From (3.15) and (3.2) we also deduce that

$$\tilde{T}_{aJ}^I \zeta_I^{(\hat{a} \hat{b}) J} = -\frac{1}{2} \epsilon_{abc} t^{bc} \delta^{\hat{a} \hat{b}} . \quad (\text{A.8})$$

Note also that we find from (3.17) that

$$\epsilon^{\hat{a} \hat{b} \hat{c}} \tilde{T}_{aJ}^I \zeta_I^{\hat{b}} \zeta^{\hat{c} J} = \epsilon^{\hat{a} \hat{b} \hat{c}} T_{aJ}^I \zeta_I^{\hat{b}} \zeta^{\hat{c} J} , \quad (\text{A.9})$$

Eq. (3.5) implies

$$(\partial_\mu \zeta_I^{\hat{a}}) \eta^{IJ} \zeta_J^{\hat{b}} = 0 , \quad (\text{A.10})$$

which means that over four-dimensional spacetime, the $J^{\hat{a}}$ do not rotate into each other and therefore really move in $\frac{SO(3,n)}{SO(3) \times SO(n)}$. Note that from (2.11) and (3.3) we also find

$$(I^{\hat{a}})_\beta^\alpha = e^{\rho_4/2} \zeta_I^{\hat{a}} \omega_{\alpha\beta}^I . \quad (\text{A.11})$$

The decomposition of ω^I into representations of $SU(2)$ reads

$$\omega^I = \zeta^{\hat{a} I} \zeta_J^{\hat{a}} \omega^J + P_J^I \omega^J , \quad (\text{A.12})$$

where $P_J^I = \delta_J^I - \zeta^{\hat{a} I} \zeta_J^{\hat{a}}$. The latter term in (A.12) is invariant under the $I^{\hat{a}}$. We hence find

$$\frac{1}{2} (I^{\hat{a}})_\beta^\alpha (\omega^I)_{\beta\gamma} e^\alpha \wedge e^\gamma = e^{-\rho_4/2} \epsilon^{\hat{a} \hat{b} \hat{c}} \zeta^{\hat{b} I} J^{\hat{c}} , \quad (\text{A.13})$$

From (A.10) and from the definition of P_J^I it follows that for the derivative of an $SU(2)$ singlet it holds that

$$dM^I = P_J^I dM^J - \zeta^{\hat{a} I} \partial_\mu \zeta_J^{\hat{a}} M^J e^\mu , \quad (\text{A.14})$$

where we used that $M^I = P_J^I M^J$.

From the definition of the Hodge dual

$$(*_4 \omega^I)_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \omega_{\gamma\delta}^I = H_J^I \omega_{\alpha\beta}^J, \quad (\text{A.15})$$

and the decomposition (A.12) we find

$$\frac{1}{4}(\omega^I)_{\alpha\beta}(\omega^J)_{\alpha\beta} = e^{-\rho_4} H^{IJ}, \quad (\text{A.16})$$

which implies

$$(\omega^I)_{\alpha\beta} (I^{\hat{a}})_{\beta}^{\alpha} = 4e^{-\rho_4/2} \zeta^{\hat{a}I}. \quad (\text{A.17})$$

From (A.16) we can deduce

$$\omega_{\gamma(\alpha}^I \omega_{\beta)\gamma}^J = -e^{-\rho_4} H^{IJ} \delta_{\alpha\beta} + 2e^{-\rho_4/2} \zeta^{\hat{a}(I} P_K^J) \omega_{\alpha\gamma}^K (I^{\hat{a}})_{\beta}^{\gamma}. \quad (\text{A.18})$$

A.2 Connection

In this section we want to compute the connection from the three Maurer-Cartan equations (2.20), using (3.3) and (3.15). From the first equation in (2.20) we see that both $\lambda_{\alpha}^{\mu}(\hat{e}_{\beta})$ and $\gamma_a^{\mu}(\hat{K}_b)$ are symmetric in their lower indices.

Now let us first solve the second equation. From the explicit form of K^a in (3.3) we find with help of (3.15)

$$\begin{aligned} dK^a = & \frac{1}{3} D\rho_3 \wedge K^a + (k^{-1})_b^c (Dk_c^a) \wedge K^b + e^{\rho_3/3} k_b^a DG^b \\ & + \frac{1}{2} e^{-\rho_3/3} k_b^a t^{bc} k_c^d \epsilon_{def} K^e \wedge K^f + e^{\rho_3/3} k_b^a t_I^b \omega^I. \end{aligned} \quad (\text{A.19})$$

where the explicit form of DG^a can be found in (A.6) and we defined

$$\begin{aligned} D\rho_3 = & d\rho_3 - \epsilon_{abc} t^{bc} G^a, \\ Dk_b^a = & dk_b^a - (k_c^a \epsilon_{bde} - \frac{1}{3} k_b^a \epsilon_{cde}) t^{cd} G^e. \end{aligned} \quad (\text{A.20})$$

Comparison with (2.20) gives for the connection components

$$\begin{aligned} \tau_{\alpha}^a = & \frac{1}{2} e^{\rho_3/3} k_b^a t_I^b \omega_{\alpha\beta}^I e^{\beta} + \tau_0^a e^{\alpha} + \frac{1}{2} (\tau_b^a)_{\alpha\gamma} (I^{\hat{b}})_{\beta}^{\gamma} e^{\beta}, \\ \gamma_{\mu}^a = & e^{\rho_3/3} k_b^a D_{[\mu} G_{\nu]}^b e^{\nu} + \gamma_{\mu\nu}^a e^{\nu} + \frac{1}{2} [(k^{-1})_b^c D_{\mu} k_c^a + (k^{-1})_a^c D_{\mu} k_c^b] K^b + \frac{1}{3} (D_{\mu} \rho_3) K^a, \\ \phi^a = & -e^{-\rho_3/3} (k_c^a k_d^b t^{cd} - \frac{1}{2} \delta^{ab} k_c^e k_d^e t^{cd}) K^b - \frac{1}{2} \epsilon^{abc} (k^{-1})_b^d Dk_d^c, \end{aligned} \quad (\text{A.21})$$

where $\gamma_{\mu\nu}^a$ is symmetric in its lower indices. Note that we used (2.23) for the decomposition of τ_{α}^a .

If we use the explicit expressions for the $J^{\hat{a}}$ given in (3.3) and (2.11), the third Maurer-Cartan equation in (2.20) implies with (3.15)

$$\begin{aligned} & \frac{1}{2} d\rho_4 \wedge J^{\hat{a}} + e^{\rho_4/2} d\zeta_I^{\hat{a}} \wedge \omega^I + e^{-\rho_3/3} e^{\rho_4/2} \zeta_I^{\hat{a}} \tilde{T}_{bJ}^I (k^{-1})_a^b K^a \wedge \omega^J - e^{\rho_4/2} \zeta_I^{\hat{a}} \tilde{T}_{bJ}^I G^b \wedge \omega^J \\ = & -2\epsilon^{\hat{a}\hat{b}\hat{c}} \Psi^{\hat{b}} \wedge J^{\hat{c}} + K^a \wedge (I^{\hat{a}})_{\beta}^{\alpha} \tau_a^{\alpha} \wedge e^{\beta} + e^{\mu} \wedge (I^{\hat{a}})_{\beta}^{\alpha} \lambda_{\mu}^{\alpha} \wedge e^{\beta}. \end{aligned} \quad (\text{A.22})$$

This can be solved for the connection components as

$$\begin{aligned}
\tau_\alpha^a &= \frac{1}{2}e^{\rho_3/3}k_b^a t_I^b \omega_{\alpha\beta}^I e^\beta - \frac{1}{4}e^{-\rho_3/3}(k^{-1})_a^b \epsilon_{bcd} t^{cd} e^\alpha + \frac{1}{2}e^{-\rho_3/3}e^{\rho_4/2}(k^{-1})_a^b \zeta_I^{\hat{a}} T_{bJ}^I P_K^J \omega_{\alpha\gamma}^K (I^{\hat{a}})_\beta^\gamma e^\beta , \\
\Psi^{\hat{a}} &= \frac{1}{4}e^{-\rho_3/3}\epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{b}} \zeta^{\hat{c}J} T_{bJ}^I (k^{-1})_a^b K^a - \frac{1}{4}e^{\rho_3/3}e^{-\rho_4/2}\zeta^{\hat{a}} I t_I^b k_b^a K^a - \frac{1}{4}\epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{b}} \zeta^{\hat{c}J} T_{bJ}^I G^b , \\
\lambda_\mu^\alpha &= \frac{1}{4}(D_\mu \rho_4) e^\alpha - \frac{1}{2}e^{\rho_4/2}(D_\mu \zeta_I^{\hat{a}}) \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma e^\beta ,
\end{aligned} \tag{A.23}$$

where we defined the covariant derivatives

$$\begin{aligned}
D\rho_4 &= d\rho_4 + \epsilon_{abc} t^{bc} G^a , \\
D\zeta_I^{\hat{a}} &= d\zeta_I^{\hat{a}} - \zeta_I^{\hat{a}} T_{bJ}^I (\delta_K^J - \zeta_K^{\hat{b}} \zeta^{\hat{b}J}) G^b .
\end{aligned} \tag{A.24}$$

and the projector $P_J^I = (\delta_J^I - \zeta^{\hat{b}I} \zeta_{\hat{b}J})$. Finally, we solve for the first Maurer-Cartan equation in (2.20), by using the explicit form of λ_μ^α and γ_μ^a , given in (A.21) and (A.23). The result is

$$\begin{aligned}
\gamma_\mu^a &= e^{\rho_3/3} k_b^a D_{[\mu} G_{\nu]}^b e^\nu + \frac{1}{2}((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) K^b + \frac{1}{3}(D_\mu \rho_3) K^a , \\
\omega_\nu^\mu &= \hat{\omega}_\nu^\mu - e^{\rho_3/3} D_{[\mu} G_{\nu]}^b k_b^a K^a ,
\end{aligned} \tag{A.25}$$

where $\hat{\omega}$ is the four-dimensional connection.

In total, we find for the components of the connection

$$\begin{aligned}
\tau_\alpha^a &= \frac{1}{2}e^{\rho_3/3}k_b^a t_I^b \omega_{\alpha\beta}^I e^\beta - \frac{1}{4}e^{-\rho_3/3}(k^{-1})_a^b \epsilon_{bcd} t^{cd} e^\alpha + \frac{1}{2}e^{-\rho_3/3}e^{\rho_4/2}(k^{-1})_a^b \zeta_I^{\hat{a}} T_{bJ}^I P_K^J \omega_{\alpha\gamma}^K (I^{\hat{a}})_\beta^\gamma e^\beta , \\
\gamma_\mu^a &= e^{\rho_3/3} k_b^a D_{[\mu} G_{\nu]}^b e^\nu + \frac{1}{2}((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) K^b + \frac{1}{3}(D_\mu \rho_3) K^a , \\
\phi^a &= -e^{-\rho_3/3}(k_c^a k_d^b t^{cd} - \frac{1}{2}\delta^{ab} k_c^e k_d^e t^{cd}) K^b - \frac{1}{2}\epsilon^{abc}(k^{-1})_b^d D k_d^c , \\
\Psi^{\hat{a}} &= \frac{1}{4}e^{-\rho_3/3}\epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{b}} \zeta^{\hat{c}J} T_{bJ}^I (k^{-1})_a^b K^a - \frac{1}{4}e^{\rho_3/3}e^{-\rho_4/2}\zeta^{\hat{a}} I t_I^b k_b^a K^a - \frac{1}{4}\epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{b}} \zeta^{\hat{c}J} T_{bJ}^I G^b , \\
\lambda_\mu^\alpha &= \frac{1}{4}D_\mu \rho_4 e^\alpha - \frac{1}{2}e^{\rho_4/2}D_\mu \zeta_I^{\hat{a}} \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma e^\beta , \\
\omega_\nu^\mu &= \hat{\omega}_\nu^\mu - e^{\rho_3/3} D_{[\mu} G_{\nu]}^b k_b^a K^a ,
\end{aligned} \tag{A.26}$$

with the covariant derivatives defined by

$$\begin{aligned}
DG^a &= dG^a + \frac{1}{2}t^{ab} \epsilon_{bcd} G^c \wedge G^d , \\
D\rho_3 &= d\rho_3 - \epsilon_{abc} t^{bc} G^a , \\
D\rho_4 &= d\rho_4 + \epsilon_{abc} t^{bc} G^a , \\
Dk_b^a &= dk_b^a - (k_c^a \epsilon_{bde} - \frac{1}{3}k_b^a \epsilon_{cde}) t^{cd} G^e , \\
D\zeta_I^{\hat{a}} &= d\zeta_I^{\hat{a}} - \zeta_K^{\hat{a}} T_{bJ}^K P_I^J G^b .
\end{aligned} \tag{A.27}$$

Note that the scalar $\rho_3 + \rho_4$ is ungauged.

In the next section, we compute the Ricci curvature from the Levi-Civita connection.

For that we will also need the differential identities

$$\begin{aligned}
d\omega_{\alpha\beta}^I(\hat{e}_\mu) &= \frac{1}{2}G_\mu^a\epsilon_{abc}t^{bc}\omega_{\alpha\beta}^I - T_{aJ}^IG_\mu^a\omega_{\alpha\beta}^J - \frac{1}{2}D_\mu\rho_4\omega_{\alpha\beta}^I + e^{-\rho_4/2}D_\mu\zeta^{\hat{a}I}(I^{\hat{a}})_\beta^\alpha \\
&\quad - \zeta^{\hat{a}I}D_\mu\zeta_J^{\hat{a}}\omega_{\alpha\beta}^J + e^{-\rho_4/2}\zeta^{\hat{a}I}\zeta_J^{\hat{a}}T_{aK}^JG_\mu^a\zeta^{\hat{b}K}(I^{\hat{b}})_\beta^\alpha, \\
d\omega_{\alpha\beta}^I(\hat{K}_a) &= e^{-\rho_3/3}(k^{-1})_a^bP_K^IT_{bL}^KP_J^L\omega_{\alpha\beta}^J - e^{\rho_3/3}k_b^at_J^bP_K^J\omega_{\gamma[\alpha}\omega_{\beta]\gamma}^K, \\
\nabla_\gamma^\theta\omega_{\alpha\beta}^I &= 0.
\end{aligned} \tag{A.28}$$

that can be computed with help of (2.20), (3.15) and (A.26). Note that ∇^θ denotes the $SU(2)$ connection.

A.3 Ricci curvature

The components of the Ricci curvature are given by

$$\begin{aligned}
\text{Ric}_{\mu\nu} &= R_{\mu\lambda\nu}^\lambda + R_{\mu a\nu}^a + R_{\mu\alpha\nu}^\alpha, \\
\text{Ric}_{\mu a} &= R_{\mu\nu a}^\nu + R_{\mu b a}^b + R_{\mu\alpha a}^\alpha, \\
\text{Ric}_{\mu\alpha} &= R_{\mu\nu\alpha}^\nu + R_{\mu a\alpha}^a + R_{\mu\beta\alpha}^\beta, \\
\text{Ric}_{ab} &= R_{a\mu b}^\mu + R_{a c b}^c + R_{a\alpha b}^\alpha, \\
\text{Ric}_{a\alpha} &= R_{a\mu\alpha}^\mu + R_{a b\alpha}^b + R_{a\beta\alpha}^\beta, \\
\text{Ric}_{\alpha\beta} &= R_{\alpha\mu\beta}^\mu + R_{\alpha c\beta}^c + R_{\alpha\gamma\beta}^\gamma.
\end{aligned} \tag{A.29}$$

Let us now compute these components one by one. We start with $\text{Ric}_{\mu\nu}$. We compute

$$\begin{aligned}
R_{\mu\lambda\nu}^\lambda &= \hat{\text{Ric}}_{\mu\nu} - 3e^{2\rho_3/3}g_{3ab}\eta^{\lambda\kappa}D_{[\mu}G_{\lambda]}^aD_{[\nu}G_{\kappa]}^b, \\
R_{\mu a\nu}^a &= -\nabla_\nu(D_\mu\rho_3) - \frac{1}{3}(D_\mu\rho_3)(D_\nu\rho_3) - \frac{1}{2}\text{tr}[k^{-1} \cdot (D_\mu k) \cdot k^{-1} \cdot (D_\nu k)] \\
&\quad - \frac{1}{2}\text{tr}[(D_\mu k)^T \cdot (g_3^{-1}) \cdot (D_\nu k)] + e^{2\rho_3/3}g_{3ab}\eta^{\lambda\kappa}D_{[\mu}G_{\lambda]}^aD_{[\nu}G_{\kappa]}^b, \\
R_{\mu\alpha\nu}^\alpha &= -\nabla_\nu(D_\mu\rho_4) - \frac{1}{4}(D_\mu\rho_4)(D_\nu\rho_4) + (D_\mu\zeta_I^{\hat{a}})(D_\nu\zeta^{\hat{a}I}).
\end{aligned} \tag{A.30}$$

This gives the external Ricci curvature

$$\begin{aligned}
\text{Ric}_{\mu\nu} &= \hat{\text{Ric}}_{\mu\nu} - \nabla_\nu(D_\mu\rho_4) - \frac{1}{4}(D_\mu\rho_4)(D_\nu\rho_4) - \nabla_\nu(D_\mu\rho_3) - \frac{1}{3}(D_\mu\rho_3)(D_\nu\rho_3) \\
&\quad - \frac{1}{2}\text{tr}[k^{-1} \cdot (D_\mu k) \cdot k^{-1} \cdot (D_\nu k)] - \frac{1}{2}\text{tr}[(D_\mu k)^T \cdot (g_3^{-1}) \cdot (D_\nu k)] \\
&\quad + (D_\mu\zeta_I^{\hat{a}})(D_\nu\zeta^{\hat{a}I}) - 2e^{2\rho_3/3}g_{3ab}\hat{g}^{\lambda\kappa}D_{[\mu}G_{\lambda]}^aD_{[\nu}G_{\kappa]}^b.
\end{aligned} \tag{A.31}$$

For $\text{Ric}_{\mu a}$ we compute

$$\begin{aligned}
R_{\mu\nu a}^\nu &= -\nabla^\nu (e^{\rho_3/3} k_b^a D_{[\nu} G_{\mu]}^b) - e^{\rho_3/3} D_{[\mu} G_{\nu]}^b \left(\frac{2}{3} k_b^a D^\nu \rho_3 + \frac{3}{2} D^\nu k_b^a + \frac{1}{2} (k^{-1})_a^c D^\nu k_c^b \right), \\
R_{\mu b a}^b &= -\frac{2}{3} e^{\rho_3/3} D_{[\mu} G_{\nu]}^b k_b^a (D^\nu \rho_3) + \frac{1}{2} e^{\rho_3/3} D_{[\mu} G_{\nu]}^b (D^\nu k_b^a + (k^{-1})_a^c (D^\nu k_c^b)) \\
&\quad + \frac{1}{2} e^{-\rho_3/3} \epsilon_{abc} k_d^b t^{ed} (D_\mu k_e^c + k_e^f (D_\mu k_g^f) (k^{-1})_c^g) \\
&\quad - \frac{1}{2} e^{-\rho_3/3} (k^{-1})_b^c \epsilon_{cde} t^{de} ((k^{-1})_a^g (D_\mu k_g^b) + (k^{-1})_b^g (D_\mu k_g^a)), \\
R_{\mu\alpha a}^\alpha &= \frac{1}{4} e^{-\rho_3/3} (D_\mu \rho_4) (k^{-1})_a^b \epsilon_{bcd} t^{cd} + e^{-\rho_3/3} (k^{-1})_a^b T_{bJ}^I \zeta_I^{\hat{a}} (D_\mu \zeta^{\hat{a}J}) - e^{\rho_3/3} (D^\nu \rho_4) k_b^a D_{[\mu} G_{\nu]}^b \\
&\quad - e^{-\rho_3/3} (k^{-1})_b^c \epsilon_{cde} t^{de} \left(\frac{1}{3} (D_\mu \rho_3) \delta_b^a + \frac{1}{2} ((k^{-1})_b^d D_\mu k_d^a + (k^{-1})_a^d D_\mu k_d^b) \right), \tag{A.32}
\end{aligned}$$

and therefore find

$$\begin{aligned}
\text{Ric}_{\mu a} &= -e^{\rho_3/3} D_{[\mu} G_{\nu]}^b \left(\frac{4}{3} k_b^a D^\nu \rho_3 + k_b^a D^\nu \rho_4 + 2 D^\nu k_b^a + (k^{-1})_a^c D^\nu k_c^b \right) \\
&\quad + e^{-\rho_3/3} (k^{-1})_a^b T_{bJ}^I \zeta_I^{\hat{a}} (D_\mu \zeta^{\hat{a}J}) - \frac{1}{2} e^{-\rho_3/3} \epsilon_{abc} k_d^b t^{ed} (D_\mu k_e^c + k_e^f (D_\mu k_g^f) (k^{-1})_c^g) \\
&\quad - \nabla^\nu (e^{\rho_3/3} k_b^a D_{[\nu} G_{\mu]}^b) + e^{-\rho_3/3} (k^{-1})_a^b \epsilon_{bcd} t^{cd} \left(\frac{1}{4} D_\mu \rho_4 - \frac{1}{3} D_\mu \rho_3 \right). \tag{A.33}
\end{aligned}$$

Next we find

$$\begin{aligned}
R_{\mu\nu\alpha}^\nu &= 0, \\
R_{\mu a\alpha}^a &= 0, \\
R_{\mu\beta\alpha}^\beta &= 0,
\end{aligned} \tag{A.34}$$

where we used in the last equation that $d\omega^I = \nabla(\frac{1}{2}\omega_{\beta\alpha}^I e^\beta) \wedge e^\alpha$. Therefore we find

$$\text{Ric}_{\mu\alpha} = 0. \tag{A.35}$$

Let us now compute the internal component Ric_{ab} . We find

$$\begin{aligned}
R_{a\mu b}^\mu &= -\frac{1}{3} \nabla^\mu D_\mu \rho_3 \delta_{ab} - \frac{1}{2} \nabla^\mu ((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) - \frac{1}{9} (D^\mu \rho_3) (D_\mu \rho_3) \delta_{ab} \\
&\quad + e^{2\rho_3/3} D_{[\mu} G_{\nu]}^c k_c^a D^\mu G^{d\nu} k_d^b - \frac{1}{3} (D^\mu \rho_3) ((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) \\
&\quad + \frac{1}{4} (D_\mu k_c^a) g_3^{cd} (D^\mu k_d^b) - \frac{3}{4} (k^{-1})_a^c (D_\mu k_c^e) (D^\mu k_d^e) (k^{-1})_b^d \\
&\quad - \frac{1}{2} [k^{-1} \cdot (D^\mu k) \cdot k^{-1} \cdot D_\mu k]_{(ab)}, \\
R_{acb}^c &= -e^{-2\rho_3/3} (\epsilon_{acd} \epsilon_{bef} (k \cdot t \cdot k^T)^{de} (k \cdot t \cdot k^T)^{fc} + 2(k \cdot t_A \cdot g_3 \cdot t \cdot k^T)^{ab} \\
&\quad - 2|t|(k \cdot t \cdot k^T)^{(ab)} + \frac{1}{2} \delta^{ab} |t|^2) - \frac{1}{6} (D^\mu \rho_3) ((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) \\
&\quad + \frac{1}{4} ((k^{-1})_c^d D_\mu k_d^a + (k^{-1})_a^d D_\mu k_d^c) ((k^{-1})_b^e D_\mu k_e^c + (k^{-1})_c^e D_\mu k_e^b) \\
&\quad - \frac{2}{9} (D^\mu \rho_3) (D_\mu \rho_3) \delta^{ab}, \\
R_{aab}^\alpha &= e^{2\rho_3/3} e^{-\rho_4} k_c^a t_I^c H^{IJ} t_J^d k_d^b - \frac{1}{3} e^{-2\rho_3/3} (k^{-1})_a^c (k^{-1})_b^d \zeta_I^{\hat{a}} \zeta_J^{\hat{b}} T_{cK}^I P^{KL} T_{dL}^J \\
&\quad - \frac{1}{4} e^{-2\rho_3/3} \epsilon_{acd} (k \cdot t \cdot k^T)^{cd} \epsilon_{bef} (k \cdot t \cdot k^T)^{ef} + 2e^{-2\rho_3/3} (k \cdot t_A \cdot g_3 \cdot t \cdot k^T)^{ab} \\
&\quad - \frac{1}{2} D^\mu \rho_4 [((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) + \frac{2}{3} D_\mu \rho_3 \delta^{ab}],
\end{aligned} \tag{A.36}$$

where we defined $g_{3ab} = k_a^c k_b^c$, $|t| = g_{3ab} t^{ab}$ and $t_A = \frac{1}{2}(t - t^T)$. Then we get

$$\begin{aligned}
\text{Ric}_{ab} = & -\frac{1}{3}\nabla^\mu D_\mu \rho_3 \delta_{ab} - \frac{1}{2}\nabla^\mu ((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) \\
& + e^{2\rho_3/3} D_{[\mu} G_{\nu]}^c k_c^a D^\mu G^{d\nu} k_d^b + \frac{1}{2}(D_\mu k_c^a) g_3^{cd} (D^\mu k_d^b) - \frac{1}{2}(k^{-1})_a^c (D_\mu k_c^e) (D^\mu k_d^e) (k^{-1})_b^d \\
& - (D^\mu \rho_3 + D^\mu \rho_4) [\frac{1}{2}((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) + \frac{1}{3} D_\mu \rho_3 \delta^{ab}] \\
& - e^{-2\rho_3/3} (\epsilon_{acd} \epsilon_{bef} (k \cdot t \cdot k^T)^{de} (k \cdot t \cdot k^T)^{fc} - 2|t|(k \cdot t \cdot k^T)^{(ab)} + \frac{1}{2} \delta^{ab} |t|^2) \\
& - \frac{1}{4} e^{-2\rho_3/3} \epsilon_{acd} (k \cdot t \cdot k^T)^{cd} \epsilon_{bef} (k \cdot t \cdot k^T)^{ef} + e^{2\rho_3/3} e^{-\rho_4} k_c^a t_I^c H^I J t_J^d k_d^b \\
& - \frac{1}{3} e^{-2\rho_3/3} (k^{-1})_a^c (k^{-1})_b^d \zeta_I^{\hat{a}} \zeta_J^{\hat{a}} T_{cK}^I P^{KL} T_{dL}^J .
\end{aligned} \tag{A.37}$$

Let us now compute the internal component $\text{Ric}_{a\alpha}$. We find

$$\begin{aligned}
R_{a\mu\alpha}^\mu &= 0 , \\
R_{ab\alpha}^b &= 0 , \\
R_{a\beta\alpha}^\beta &= \nabla^\theta \tau_a^\beta (\hat{e}_\beta, \hat{e}_\alpha) = 0 ,
\end{aligned} \tag{A.38}$$

so that

$$\text{Ric}_{a\alpha} = 0 . \tag{A.39}$$

Let us now compute the internal component $\text{Ric}_{\alpha\beta}$. We find

$$\begin{aligned}
R_{\alpha\mu\beta}^\mu &= -\frac{1}{4}\nabla^\mu (D_\mu \rho_4) \delta_{\alpha\beta} + \frac{1}{2}\nabla^\mu (e^{\rho_4/2} D_\mu \zeta_I^{\hat{a}}) P_J^I \omega_{\alpha\gamma}^J (I^{\hat{a}})_\beta^\gamma \\
& - \frac{1}{16} D_\mu \rho_4 D^\mu \rho_4 \delta_{\alpha\beta} - \frac{1}{4} D^\mu \zeta_I^{\hat{a}} D_\mu \zeta_I^{\hat{a}} \delta_{\alpha\beta} \\
& + \frac{1}{4} e^{\rho_4/2} D^\mu \zeta_I^{\hat{a}} G_\mu^a \epsilon_{abc} t^{bc} \omega_{\alpha\gamma}^I (I^{\hat{b}})_\beta^\gamma - \frac{1}{2} e^{\rho_4/2} D^\mu \zeta_I^{\hat{a}} T_{aJ}^I G_\mu^a P_K^J \omega_{\alpha\gamma}^K (I^{\hat{b}})_\beta^\gamma \\
& + \frac{1}{4} e^{\rho_4/2} D^\mu \rho_4 D_\mu \zeta_I^{\hat{a}} \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma + \frac{1}{2} e^{\rho_4/2} D^\mu \zeta_I^{\hat{a}} T_{aK}^J G_\mu^a \zeta_J^{\hat{b}} \omega_{\alpha\gamma}^I (I^{\hat{b}})_\beta^\gamma , \\
R_{\alpha\alpha\beta}^a &= -\frac{1}{2} D^\mu \rho_3 (\frac{1}{2} D_\mu \rho_4 \delta_{\alpha\beta} - e^{\rho_4/2} D_\mu \zeta_I^{\hat{a}} \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma) \\
& + e^{\rho_3/3} k_b^a t_I^b \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma \Psi^{\hat{a}} (\hat{K}_a) - \tau_\alpha^a (\hat{e}_\gamma) \tau_\gamma^a (\hat{e}_\beta) - e^{-\rho_3/3} (k^{-1})_a^b \epsilon_{bcd} t^{cd} \tau_\alpha^a (\hat{e}_\beta) \\
& + \frac{1}{2} e^{-2\rho_3/3} e^{\rho_4/2} g_3^{-1ab} \zeta_I^{\hat{a}} T_{aK}^I P_L^K T_{bM}^L P_J^M \omega_{\alpha\gamma}^J (I^{\hat{a}})_\beta^\gamma \\
& - \frac{1}{2} e^{\rho_4/2} t_I^a P_J^I \zeta_K^{\hat{a}} T_{aL}^K P_M^L \omega_{\alpha\rho}^J \omega_{\rho\gamma}^M (I^{\hat{a}})_\beta^\gamma , \\
R_{\alpha\gamma\beta}^\gamma &= \text{Ric}_{\alpha\beta}^\theta - \frac{3}{16} D^\mu \rho_4 D_\mu \rho_4 \delta_{\alpha\beta} + \frac{1}{4} D^\mu \zeta_I^{\hat{a}} D_\mu \zeta_I^{\hat{a}} \delta_{\alpha\beta} + \frac{1}{4} e^{\rho_4/2} D^\mu \rho_4 D_\mu \zeta_I^{\hat{a}} \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma \\
& - e^{\rho_3/3} k_b^a t_I^b \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma \Psi^{\hat{a}} (\hat{K}_a) + \tau_\alpha^a (\hat{e}_\gamma) \tau_\gamma^a (\hat{e}_\beta) + e^{-\rho_3/3} (k^{-1})_a^b \epsilon_{bcd} t^{cd} \tau_\alpha^a (\hat{e}_\beta) ,
\end{aligned} \tag{A.40}$$

where $\text{Ric}_{\alpha\beta}^\theta$ is the Ricci curvature of the $SU(2)$ connection θ . We can compute $\text{Ric}_{\alpha\beta}^\theta$ by taking the $SU(2)$ -covariant derivative of the torsion of the connection θ [1]

$$\text{Ric}_{\alpha\beta}^\theta = \frac{1}{6} (dT^\alpha + \theta_\lambda^\alpha \wedge T^\lambda) (\hat{e}_\gamma, \hat{e}_\delta, \hat{e}_\rho) \epsilon^{\gamma\delta\rho\beta} . \tag{A.41}$$

From (2.20) we find that the component T^α of the torsion tensor of θ is

$$\begin{aligned}
T^\alpha &= de^\alpha + \theta_\beta^\alpha \wedge e^\beta \\
&= -\psi^{\hat{a}} \wedge (I^{\hat{a}})_\beta^\alpha e^\beta + K^a \wedge \tau_a^\alpha + e^\mu \wedge \lambda_\mu^\alpha .
\end{aligned} \tag{A.42}$$

From this we find for the $SU(2)$ Ricci curvature

$$\begin{aligned}
\text{Ric}_{\alpha\beta}^{\theta} = & -\frac{1}{2}e^{2\rho_3/3}e^{-\rho_4}g_{3ab}t_I^aH^{IJ}t_J^b\delta_{\alpha\beta} - \frac{1}{2}e^{-\rho_4/2}\epsilon^{\hat{a}\hat{b}\hat{c}}\zeta_I^{\hat{a}}\zeta_J^{\hat{b}}\zeta^{\hat{c}K}t_I^aT_{aK}^J\delta_{\alpha\beta} \\
& - \frac{1}{2}e^{2\rho_3/3}e^{-\rho_4/2}g_{3ab}\zeta^{\hat{a}I}t_I^aP_K^JP_K^J\omega_{\alpha\gamma}^K(I^{\hat{a}})_{\beta}^{\gamma} + \frac{1}{2}e^{\rho_4/2}t_I^aP_J^I\zeta_K^{\hat{a}}T_{aL}^KP_M^L\omega_{\alpha\rho}^{[J}\omega_{\rho\gamma}^{M]}(I^{\hat{a}})_{\beta}^{\gamma} \\
& - \frac{1}{2}\epsilon^{\hat{a}\hat{b}\hat{c}}\zeta^{\hat{a}I}\zeta_J^{\hat{b}}t_I^aT_{aK}^JP_L^K\omega_{\alpha\gamma}^L(I^{\hat{c}})_{\beta}^{\gamma} .
\end{aligned} \tag{A.43}$$

Now we are able to determine $\text{Ric}_{\alpha\beta}$. We find

$$\begin{aligned}
\text{Ric}_{\alpha\beta} = & -\frac{1}{4}\nabla^{\mu}(D_{\mu}\rho_4)\delta_{\alpha\beta} - \frac{1}{4}D_{\mu}\rho_4D^{\mu}\rho_4\delta_{\alpha\beta} - \frac{1}{4}D^{\mu}\rho_3D_{\mu}\rho_4\delta_{\alpha\beta} \\
& - \frac{1}{2}e^{2\rho_3/3}e^{-\rho_4}g_{3ab}t_I^aH^{IJ}t_J^b\delta_{\alpha\beta} - \frac{1}{2}e^{-\rho_4/2}\epsilon^{\hat{a}\hat{b}\hat{c}}\zeta_I^{\hat{a}}\zeta_J^{\hat{b}}\zeta^{\hat{c}K}t_I^aT_{aK}^J\delta_{\alpha\beta} \\
& + \frac{1}{2}\nabla^{\mu}(e^{\rho_4/2}D_{\mu}\zeta_I^{\hat{a}})\omega_{\alpha\gamma}^I(I^{\hat{a}})_{\beta}^{\gamma} + \frac{1}{2}e^{\rho_4/2}D^{\mu}(\rho_4 + \rho_3)D_{\mu}\zeta_I^{\hat{a}}\omega_{\alpha\gamma}^I(I^{\hat{a}})_{\beta}^{\gamma} \\
& + \frac{1}{4}e^{\rho_4/2}D^{\mu}\zeta_I^{\hat{a}}G_{\mu}^a\epsilon_{abc}t^{bc}\omega_{\alpha\gamma}^I(I^{\hat{b}})_{\beta}^{\gamma} - \frac{1}{2}e^{\rho_4/2}D^{\mu}\zeta_I^{\hat{a}}T_{aJ}^IG_{\mu}^aP_K^JP_K^J\omega_{\alpha\gamma}^K(I^{\hat{b}})_{\beta}^{\gamma} \\
& + \frac{1}{2}e^{\rho_4/2}D^{\mu}\zeta_I^{\hat{a}}T_{aK}^JG_{\mu}^a\zeta_J^{\hat{b}}\zeta^{\hat{c}K}\omega_{\alpha\gamma}^I(I^{\hat{b}})_{\beta}^{\gamma} - \frac{1}{2}e^{2\rho_3/3}e^{-\rho_4/2}g_{3ab}\zeta^{\hat{a}I}t_I^aP_K^JP_K^J\omega_{\alpha\gamma}^K(I^{\hat{a}})_{\beta}^{\gamma} \\
& - \frac{1}{2}\epsilon^{\hat{a}\hat{b}\hat{c}}\zeta_I^{\hat{b}}\zeta_J^{\hat{c}}T_{aJ}^IT_{aK}^K\omega_{\alpha\gamma}^I(I^{\hat{a}})_{\beta}^{\gamma} + \frac{1}{2}e^{-2\rho_3/3}e^{\rho_4/2}g_3^{-1ab}\zeta_I^{\hat{a}}T_{aK}^IP_L^KT_{bM}^LP_J^M\omega_{\alpha\gamma}^J(I^{\hat{a}})_{\beta}^{\gamma} .
\end{aligned} \tag{A.44}$$

From (A.31), (A.37) and (A.44) we can compute the Ricci scalar. It reads

$$\begin{aligned}
r_{11} = & r_4 - 2\nabla^{\mu}(D_{\mu}\rho_4) - 2\nabla^{\mu}(D_{\mu}\rho_3) \\
& - \frac{5}{4}(D_{\mu}\rho_4)(D^{\mu}\rho_4) - \frac{4}{3}(D_{\mu}\rho_3)(D^{\mu}\rho_3) - 2D^{\mu}\rho_4D_{\mu}\rho_3 \\
& - \frac{1}{2}\text{tr}[k^{-1} \cdot (D_{\mu}k) \cdot k^{-1} \cdot (D^{\mu}k)] - \frac{1}{2}\text{tr}[(D_{\mu}k)^T \cdot (g_3^{-1}) \cdot (D^{\mu}k)] \\
& + (D_{\mu}\zeta_I^{\hat{a}})(D^{\mu}\zeta^{\hat{a}I}) - e^{2\rho_3/3}g_{3ab}D_{[\mu}G_{\nu]}^aD^{\mu}G^{b\nu} \\
& + \frac{3}{2}e^{-2\rho_3/3}(g_{3ab}t^{ab})^2 - e^{-2\rho_3/3}g_{3ab}g_{3cd}t^{(ac)}t^{(bd)} - \frac{3}{4}e^{-2\rho_3/3}g_3^{ab}\epsilon_{acd}\epsilon_{bef}t^{cd}t^{ef} \\
& - e^{2\rho_3/3}e^{-\rho_4}g_{3ab}t_I^aH^{IJ}t_J^b - \frac{1}{3}e^{-2\rho_3/3}g_3^{ab}\zeta_I^{\hat{a}}\zeta_J^{\hat{b}}T_{aK}^IP_{bL}^KT_{bL}^J \\
& - 2e^{-\rho_4/2}\epsilon^{\hat{a}\hat{b}\hat{c}}\zeta^{\hat{a}I}\zeta_J^{\hat{b}}\zeta^{\hat{c}K}t_I^aT_{aK}^J .
\end{aligned} \tag{A.45}$$

A.4 Ricci curvature in the Einstein frame

In order to define the four-dimensional theory in the Einstein frame, we have to perform the Weyl rescaling (3.29). As the scalar fields only depend on four-dimensional spacetime, this mostly affects the Ricci curvature component $\text{Ric}_{\mu\nu}$, given in (A.31). It reads in the four-dimensional Einstein frame

$$\begin{aligned}
\text{Ric}_{\mu\nu} = & \tilde{\text{Ric}}_{\mu\nu} - \frac{3}{4}(D_{\mu}\phi)(D_{\nu}\phi) - \frac{1}{2}(D_{\mu}\rho_3)(D_{\nu}\rho_3) + \frac{1}{2}\tilde{g}_{\mu\nu}\nabla^{\rho}D_{\rho}\phi + \frac{1}{6}\tilde{g}_{\mu\nu}\nabla^{\rho}D_{\rho}\rho_3 \\
& - \frac{1}{2}\text{tr}[k^{-1} \cdot (D_{\mu}k) \cdot k^{-1} \cdot (D_{\nu}k)] - \frac{1}{2}\text{tr}[(D_{\mu}k)^T \cdot (g_3^{-1}) \cdot (D_{\nu}k)] \\
& + (D_{\mu}\zeta_I^{\hat{a}})(D_{\nu}\zeta^{\hat{a}I}) - 2e^{2\rho_3/3}g_{3ab}\hat{g}^{\lambda\kappa}D_{[\mu}G_{\lambda]}^aD_{[\nu}G_{\kappa]}^b .
\end{aligned} \tag{A.46}$$

The only non-trivial off-diagonal component of the Ricci curvature is

$$\begin{aligned}\text{Ric}_{\mu a} = & -e^{-\rho_3/3}(k^{-1})_a^b \nabla^\nu (e^{\phi+\rho_3} g_{3bc} D_{[\nu} G_{\mu]}^c) \\ & + e^{-\rho_3/3}(k^{-1})_a^b T_{bJ}^I \zeta_I^{\hat{a}} (D_\mu \zeta^{\hat{a}J}) - \frac{1}{2} e^{-\rho_3/3} \epsilon_{abc} k_d^b t^{ed} (D_\mu k_e^c + k_e^f (D_\mu k_g^f) (k^{-1})_c^g) \\ & + e^{-\rho_3/3}(k^{-1})_a^b \epsilon_{bcd} t^{cd} (\frac{1}{4} D_\mu \phi - \frac{1}{2} D_\mu \rho_3) ,\end{aligned}\tag{A.47}$$

which, as we discuss in Section 3.1, corresponds to the equation of motion for G^a . For the other components of the Ricci curvature we find

$$\begin{aligned}\text{Ric}_{ab} = & -\frac{1}{3} e^{\phi+\rho_3/3} \nabla^\mu D_\mu \rho_3 \delta_{ab} - \frac{1}{2} e^{\phi+\rho_3/3} \nabla^\mu ((k^{-1})_b^c D_\mu k_c^a + (k^{-1})_a^c D_\mu k_c^b) \\ & + e^{2\phi+4\rho_3/3} D_{[\mu} G_{\nu]}^c k_c^a D^\mu G^{d\nu} k_d^b \\ & + \frac{1}{2} e^{\phi+\rho_3/3} (D_\mu k_c^a) g_3^{cd} (D^\mu k_d^b) - \frac{1}{2} e^{\phi+\rho_3/3} (k^{-1})_a^c (D_\mu k_c^e) (D^\mu k_d^e) (k^{-1})_b^d \\ & - e^{-2\rho_3/3} (\epsilon_{acd} \epsilon_{bef} (k \cdot t \cdot k^T)^{de} (k \cdot t \cdot k^T)^{fc} - 2|t| (k \cdot t \cdot k^T)^{(ab)} + \frac{1}{2} \delta^{ab} |t|^2) \\ & - \frac{1}{4} e^{-2\rho_3/3} \epsilon_{acd} (k \cdot t \cdot k^T)^{cd} \epsilon_{bef} (k \cdot t \cdot k^T)^{ef} + e^{2\rho_3/3} e^{-\rho_4} k_c^a t_I^c H^{IJ} t_J^d k_d^b \\ & - \frac{1}{3} e^{-2\rho_3/3} (k^{-1})_a^c (k^{-1})_b^d \zeta_I^{\hat{a}} \zeta_J^{\hat{b}} T_{cK}^I P^{KL} T_{dL}^J ,\end{aligned}\tag{A.48}$$

$$\begin{aligned}\text{Ric}_{\alpha\beta} = & -\frac{1}{4} e^{\phi+\rho_3/3} \nabla^\mu D_\mu (\phi - \frac{2}{3} \rho_3) \delta_{\alpha\beta} - \frac{1}{2} e^{-\phi+4\rho_3/3} g_{3ab} t_I^a H^{IJ} t_J^b \delta_{\alpha\beta} \\ & - \frac{1}{2} e^{-\phi/2+\rho_3/3} \epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{a}} \zeta_J^{\hat{b}} \zeta^{\hat{c}K} t_I^a T_{aK}^J \delta_{\alpha\beta} \\ & + \frac{1}{2} e^{3\phi/2} \nabla^\mu (D_\mu \zeta_I^{\hat{a}}) \omega_{\alpha\gamma}^I (I^{\hat{a}})_\beta^\gamma - \frac{1}{2} e^{3\phi/2} D^\mu \zeta_I^{\hat{a}} T_{aJ}^I G_\mu^a P_K^J \omega_{\alpha\gamma}^K (I^{\hat{b}})_\beta^\gamma \\ & + \frac{1}{2} e^{3\phi/2} D^\mu \zeta_I^{\hat{a}} T_{aK}^J G_\mu^a \zeta_J^{\hat{b}} \zeta^{\hat{c}K} \omega_{\alpha\gamma}^I (I^{\hat{b}})_\beta^\gamma - \frac{1}{2} e^{\rho_3} e^{-\phi/2} g_{3ab} \zeta^{\hat{a}I} t_I^a t_J^b P_K^J \omega_{\alpha\gamma}^K (I^{\hat{a}})_\beta^\gamma \\ & - \frac{1}{2} \epsilon^{\hat{a}\hat{b}\hat{c}} \zeta_I^{\hat{a}} \zeta_J^{\hat{b}} \zeta^{\hat{c}J} T_{aJ}^I t_K^a \omega_{\alpha\gamma}^K (I^{\hat{a}})_\beta^\gamma + \frac{1}{2} e^{\phi/2-\rho_3} g_3^{-1ab} \zeta_I^{\hat{a}} T_{aK}^I P_L^K T_{bM}^L P_J^M \omega_{\alpha\gamma}^J (I^{\hat{a}})_\beta^\gamma .\end{aligned}$$

Similarly, the eleven-dimensional Ricci scalar (A.45) transforms to

$$\begin{aligned}g_{\mu\nu} r_{11} = & \tilde{g}_{\mu\nu} \left(\tilde{r}_4 + \nabla^\rho D_\rho \phi + \frac{1}{3} \nabla^\rho D_\rho \rho_3 - \frac{1}{2} (D_\rho \rho_3) (D^\rho \rho_3) - \frac{3}{4} (D_\rho \phi) (D^\rho \phi) \right. \\ & - \frac{1}{4} \text{tr}[(D_\rho g_3) \cdot (g_3^{-1}) \cdot (D^\rho g_3) \cdot g_3^{-1}] \\ & + (D_\rho \zeta_I^{\hat{a}}) (D^\rho \zeta^{\hat{a}I}) - e^{\phi+\rho_3} g_{3ab} D_{[\rho} G_{\sigma]}^a D^\rho G^{b\sigma} \\ & + \frac{3}{2} e^{-\phi-\rho_3} (g_{3ab} t^{ab})^2 - e^{-\phi-\rho_3} g_{3ab} g_{3cd} t^{(ac)} t^{(bd)} \\ & - \frac{3}{4} e^{-\phi-\rho_3} g_3^{ab} \epsilon_{acd} \epsilon_{bef} t^{cd} t^{ef} - e^{-2\phi+\rho_3} g_{3ab} t_I^a H^{IJ} t_J^b \\ & \left. - \frac{1}{3} e^{-\phi-\rho_3} g_3^{ab} \zeta_I^{\hat{a}} \zeta_J^{\hat{b}} T_{aK}^I P^{KL} T_{bL}^J - 2e^{-\frac{3}{2}\phi} \epsilon^{\hat{a}\hat{b}\hat{c}} \zeta^{\hat{a}I} \zeta_J^{\hat{b}} \zeta^{\hat{c}K} t_I^a T_{aK}^J \right) .\end{aligned}\tag{A.49}$$

B Form field computations

B.1 Field dualizations

Here we now explicitly discuss the field dualizations for some of the components of the three-form field C_3 . The main reason for the dualization is exchanging the two-forms \hat{C}_a

for scalars γ^a and dualizing into a standard electric-magnetic duality frame of $N = 4$ gauged supergravity, as described in [8]. As we will see, this requires to exchange the gauge fields C^a for their magnetic duals, which we will denote by \tilde{C}_a . However, to perform the dualization in a consistent way, we also have to introduce further dual auxiliary fields: A magnetic vector field \tilde{C}_I and the two-forms \hat{C}_0 and \hat{C}_I^a . The magnetic vector field \tilde{C}_I will appear in the covariant derivative of γ^a , because \hat{C}_a appears in the covariant derivative of C_I . The new two-forms will appear in the covariant derivative of \tilde{C}_a , since C^a appears in the covariant derivatives of the scalars c_{aI} and c_0 .

We perform the field dualizations by showing that the set of Bianchi identities and equations of motions are the same, with Bianchi identities swapped for the equations of motion and vice versa. Therefore, let us start by deriving the Bianchi identities from $dG_4 = 0$. This gives for the field strengths $\hat{F}_a = D\hat{C}_a - \epsilon_{abc}C^b \wedge DG^c$, $F_I = DC_I + c_{aI}DG^a$ and $F^a = DC^a + c_0DG^a$ the identities

$$\begin{aligned} d\hat{F}_a + \epsilon_{abc}t^{dc}G^b \wedge \hat{F}_d + \epsilon_{abc}F^b \wedge DG^c &= 0, \\ dF_I - t_I^a\hat{F}_a - T_{aI}^JG^a \wedge F_J + \frac{1}{2}\epsilon_{abc}t^{bc}G^a \wedge F_I - Dc_{aI} \wedge DG^a &= 0, \\ dF^a - t^{ba}\hat{F}_b - Dc_0 \wedge DG^a + t^{ab}\epsilon_{bcd}G^c \wedge F^d - \epsilon_{bcd}t^{cd}G^b \wedge F^a &= 0. \end{aligned} \quad (B.1)$$

Furthermore, we vary the Lagrangian with respect to the fields \hat{C}_a , C^a and C_I to determine their equations of motion to be

$$\begin{aligned} 0 &= d(e^{2\phi}g_3^{ab} * \hat{F}_b) + t^{ab}\epsilon_{bcd}G^c \wedge (e^{2\phi}g^{de} * \hat{F}_e) \\ &\quad + e^{\phi-\rho_3}t^{ab}g_{3bc} * F^c + e^{\rho_3}t_I^aH^{IJ} * F_J + \frac{1}{2}\epsilon^{abc}\eta^{IJ}Dc_{bI} \wedge Dc_{cJ} \\ &\quad + f_0DG^a + (g_0 + c_{aI}\eta^{IJ}t_J^a)F^a + (g_I^a + c_0t_I^a + \epsilon^{abc}T_{bI}^Jc_{cJ} + t^{(ab)}c_{bI})\eta^{IJ}F_J, \\ 0 &= (d + \epsilon_{bcd}t^{bc}G^d \wedge)(e^{\phi-\rho_3}g_{3ae} * F^e) - \epsilon_{abc}t^{db}G^c \wedge (e^{\phi-\rho_3}g_{3de} * F^e) \\ &\quad - \epsilon_{abc}DG^b \wedge (e^{2\phi}g_3^{cd} * \hat{F}_d) + \epsilon_{abc}t^{bc}(e^{-2\rho_3} * Dc_0) - f_0\epsilon_{abc}G^b \wedge DG^c \\ &\quad - \epsilon_{abc}t_I^cH^{IJ}e^{-\phi}g^{bd} * Dc_{dJ} + (g_0 + c_{bI}\eta^{IJ}t_J^b)\hat{F}^a + \eta^{IJ}Dc_{aI} \wedge F_J, \\ 0 &= d(e^{\rho_3}H_I^J * F_J) - T_{aI}^JG^a \wedge (e^{\rho_3}H_J^K * F_K) - \frac{1}{2}\epsilon_{abc}t^{bc}G^a \wedge (e^{\rho_3}H_I^J * F_J) \\ &\quad - T_{aI}^JH_J^Kg_3^{ab}e^{-\phi} * Dc_{bK} - \frac{1}{2}\epsilon_{abc}t^{bc}H_I^Jg_3^{ad}e^{-\phi} * Dc_{dJ} \\ &\quad + (g_I^a + c_0t_I^a + \epsilon^{abc}T_{bI}^Jc_{cJ} + t^{(ab)}c_{bI})\hat{F}_a + Dc_{aI} \wedge F^a + Dc_0 \wedge F_I. \end{aligned} \quad (B.2)$$

We now want to introduce dual fields with field strengths Γ^a , \tilde{F}_a and \tilde{F}_I in such a way that on-shell the duality relations

$$\begin{aligned} \Gamma^a &= e^{2\phi}g_3^{ab} * \hat{F}_b, \\ \tilde{F}_a &= e^{\phi-\rho_3}g_{3ab} * F^b, \\ \tilde{F}_I &= e^{\rho_3}H_I^J * F_J, \end{aligned} \quad (B.3)$$

should hold. Similarly, we need to introduce two-forms \hat{C}_0 and \hat{C}_I^a that are related to the scalars c_0 and c_{aI} by similar duality relations. We also need to include seven-form flux

G_7^{flux} given in (3.11) on the internal space, which will play an important role as soon as we introduce potentials.

From (B.2) we find the Bianchi identities

$$\begin{aligned}
0 &= d\Gamma^a + t^{ab}\epsilon_{bcd}\Gamma^c \wedge G^d + t^{ab}\tilde{F}_b + t_I^a\eta^{IJ}\tilde{F}_J - \frac{1}{2}\epsilon^{abc}\eta^{IJ}Dc_{bI} \wedge Dc_{cJ} \\
&\quad + f_0DG^a + (g_0 + c_{aI}\eta^{IJ}t_J^a)F^a + (g_I^a + c_0t_I^a + \epsilon^{abc}T_{bI}^Jc_{cJ} + t^{(ab)}c_{bI})\eta^{IJ}F_J , \\
0 &= d\tilde{F}_a + \epsilon_{bcd}t^{bc}G^d \wedge \tilde{F}_a - \epsilon_{abc}t^{db}G^c \wedge \tilde{F}_d + \epsilon_{abc}\Gamma^b \wedge DG^c + \epsilon_{abc}t^{bc}\hat{F}_0 \\
&\quad - \epsilon_{abc}t_I^c\eta^{IJ}\hat{F}_J^b + (g_0 + c_{bI}\eta^{IJ}t_J^b)\hat{F}^a + \eta^{IJ}Dc_{aI} \wedge F_J - f_0\epsilon_{abc}G^b \wedge DG^c , \\
0 &= d\tilde{F}_I - T_{aI}^JG^a \wedge \tilde{F}_J - \frac{1}{2}\epsilon_{abc}t^{bc}G^a \wedge \tilde{F}_I - T_{aI}^J\hat{F}_J^a - \frac{1}{2}\epsilon_{abc}t^{bc}\hat{F}_I^a \\
&\quad + (g_I^a + c_0t_I^a + \epsilon^{abc}T_{bI}^Jc_{cJ} + t^{(ab)}c_{bI})\hat{F}_a + Dc_{aI} \wedge F^a + Dc_0 \wedge F_I .
\end{aligned} \tag{B.4}$$

Here \hat{F}_0 and \hat{F}_I^a denote the field strengths of the two-forms \hat{C}_0 and \hat{C}_I^a , respectively. The Bianchi identities are solved in terms of potentials γ^a , \tilde{C}_a and \tilde{C}_I as well as \hat{C}_0 and \hat{C}_I^a by

$$\begin{aligned}
\hat{F}_0 &= d\hat{C}_0 + \epsilon_{abc}t^{bc}G^a \wedge C_0 - \epsilon_{abc}t_I^a\eta^{IJ}G^b \wedge \hat{C}_J^c - \frac{1}{2}t_I^a\eta^{IJ}C_J \wedge \hat{C}_a + \frac{1}{2}\eta^{IJ}C_I \wedge F_J \\
&\quad - \frac{1}{6}f_0\epsilon_{abc}G^a \wedge G^b \wedge G^c , \\
\hat{F}_I^a &= d\hat{C}_I^a - T_{bI}^JG^a \wedge \hat{C}_J^b - \frac{1}{2}\epsilon_{bcd}t^{cd}G^a \wedge \hat{C}_I^b - t^{ba}C_I \wedge \hat{C}_b - C^a \wedge F_I - \epsilon^{abc}c_{bI}\hat{F}_c , \\
\tilde{F}_a &= D\tilde{C}_a - c_{aI}\eta^{IJ}F_J - \epsilon_{abc}\gamma^b DG^c + \frac{1}{2}c_{aI}\eta^{IJ}c_{bJ}DG^b , \\
\tilde{F}_I &= D\tilde{C}_I - c_0F_I - c_{aI}F^a + c_0c_{aI}DG^a , \\
\Gamma^a &= D\gamma^a + \frac{1}{2}\epsilon^{abc}\eta^{IJ}c_{bI}Dc_{cJ} ,
\end{aligned} \tag{B.5}$$

where we defined the covariant derivatives

$$\begin{aligned}
D\tilde{C}_a &= d\tilde{C}_a - \epsilon_{abc}t^{bc}\hat{C}_0 - \epsilon_{abc}t_I^b\eta^{IJ}\hat{C}_J^c - g_0\hat{C}_a - \frac{1}{2}T_{aK}^I\eta^{KJ}C_I \wedge C_J \\
&\quad + \epsilon_{bcd}t^{bc}G^d \wedge \tilde{C}_a - \epsilon_{abc}t^{db}G^c \wedge \tilde{C}_d + \frac{1}{2}f_0\epsilon_{abc}G^b \wedge G^c , \\
D\tilde{C}_I &= d\tilde{C}_I + T_{aI}^J\hat{C}_J^a + \frac{1}{2}\epsilon_{abc}t^{bc}\hat{C}_I^a - g_I^a\hat{C}_a + \frac{1}{2}t_I^a\epsilon_{abc}C^b \wedge C^c - g_I^a\epsilon_{abc}G^b \wedge C^c \\
&\quad + T_{aI}^JC^a \wedge C_J + \frac{1}{2}\epsilon_{abc}t^{bc}C^a \wedge C_I - T_{aI}^JG^a \wedge \tilde{C}_J - \frac{1}{2}\epsilon_{abc}t^{bc}G^a \wedge \tilde{C}_I , \\
D\gamma^a &= d\gamma^a - (f_0\delta_d^a + t^{ab}\epsilon_{bcd}(\gamma^c + \frac{1}{2}\epsilon^{cef}c_{eI}\eta^{IJ}c_{fJ}))G^d - t^{ab}\tilde{C}_b - t_I^a\eta^{IJ}\tilde{C}_J - g_0C^a \\
&\quad - g_I^a\eta^{IJ}C_J + c_{bI}\eta^{IJ}(t_J^{[a}C^{b]} - g_J^{[a}G^{b]}) + \frac{1}{2}\epsilon^{abc}c_{bI}\eta^{IJ}T_{cJ}^KC_K + \frac{1}{2}t^{[ab]}c_{bI}\eta^{IJ}C_J .
\end{aligned} \tag{B.6}$$

By dualizing the Bianchi identities (B.1) for \hat{C}_a and C^a we find the equations of motion for \tilde{C}_a and γ^a . They read

$$\begin{aligned}
0 &= d(e^{-2\phi}g_{3ab} * \Gamma^b) + \epsilon_{abc}t^{dc}G^b \wedge (e^{-2\phi}g_{3de} * \Gamma^e) - \epsilon_{abc}DG^b \wedge (e^{\rho_3-\phi}g_3^{cd} * \tilde{F}_d) , \\
0 &= d(e^{\rho_3-\phi}g_3^{ab} * \tilde{F}_b) - t^{ba}(e^{-2\phi}g_{3bc} * \Gamma^c) + t^{ab}\epsilon_{bcd}G^c \wedge (e^{\rho_3-\phi}g_3^{de} * \tilde{F}_e) \\
&\quad - \epsilon_{bcd}t^{cd}G^b \wedge (e^{\rho_3-\phi}g_3^{de} * \tilde{F}_e) - Dc_0 \wedge DG^a .
\end{aligned} \tag{B.7}$$

Now we are in the position to give the dual Lagrangian.

$$\begin{aligned}
S_{\text{dual}} = & -\frac{1}{4\kappa_4^2} \int \left(e^{\rho_3 - \phi} g_3^{ab} (D\tilde{C}_a - c_{aI}\eta^{IK}DC_K - \epsilon_{acd}\gamma^c DG^d - \frac{1}{2}c_{aI}\eta^{IK}c_{cK}DG^c) \right. \\
& \wedge * (D\tilde{C}_b - c_{bJ}\eta^{JL}DC_L - \epsilon_{bef}\gamma^e DG^f - \frac{1}{2}c_{bJ}\eta^{JL}c_{eL}DG^e) \\
& - e^{-2\phi} g_{3ab} (D\gamma^a + \frac{1}{2}\epsilon^{acd}c_{cI}\eta^{IJ}Dc_{dJ}) \wedge *_4 (D\gamma^b + \frac{1}{2}\epsilon^{bef}c_{eI}\eta^{IJ}Dc_{fJ}) \\
& - 2c_0 D\tilde{C}_a \wedge DG^a + g_0 (2DC^a - t^{ba}\hat{C}_b) \wedge \hat{C}_a \\
& + 2DC^a \wedge \epsilon_{abc} (t^{bc}\hat{C}_0 + t_I^b \eta^{IJ}\hat{C}_J^c) - t_I^a \eta^{IJ} (2D\tilde{C}_J - g_J^b \hat{C}_b) \wedge \hat{C}_a \\
& + \epsilon_{abc} t^{bc} \eta^{IJ} C^a \wedge C_I \wedge DC_J + t_I^a \eta^{IJ} \epsilon_{abc} C^a \wedge C^b \wedge DC_J \\
& \left. + 2\epsilon_{abc} t_I^a \eta^{IJ} \tilde{C}_J \wedge C^b \wedge DG^c + \frac{1}{6} f_0 \epsilon_{def} t^{ef} \epsilon_{abc} G^a \wedge G^b \wedge G^c \wedge C^d \right) . \tag{B.8}
\end{aligned}$$

The first two terms are the new kinetic terms that replace the ones of \hat{C}_a and C^a . The next term is a topological term to complete the equations of motion of (B.7). The remaining topological terms ensure that variation with respect to the auxiliary fields gives the duality relations (B.3) as well as⁴

$$\begin{aligned}
\hat{F}_a &= e^{-2\phi} g_{3ab} * (D\gamma^b + \frac{1}{2}\epsilon^{bcd}c_{cJ}\eta^{JK}Dc_{dK}) , \\
\hat{F}_0 &= e^{-2\rho_3} * DC_0 , \\
\hat{F}_I^a &= e^{-\phi} H_I^J g_3^{ab} * DC_{bJ} . \tag{B.9}
\end{aligned}$$

Note that γ^a cannot have a potential term since it inherits the shift symmetry from its dual tensor \hat{C}_a .

B.2 Energy-Momentum tensor

In this appendix we want to compute the eleven-dimensional energy-momentum tensor

$$T_{AB} = \frac{1}{12} (G_{ACDE} G_B^{CDE} - \frac{1}{4} g_{AB} G_{CDEF} G^{CDEF}) , \tag{B.10}$$

that appears in the Einstein field equations (5.5). For this, we use the form (3.33) and the field dualizations that have been discussed in more detail in Section 3.3 and in Appendix B.1.

First of all note that the components $T_{\mu\alpha}$ and $T_{a\alpha}$ both are identical zero, due to the absence of $SU(2)$ doublet degrees of freedom in our ansatz. This fits nicely together with (A.35) and (A.39). For the remaining components, let us first compute $\tilde{T}_{AB} =$

⁴Note that these equations only arise if the corresponding charges are non-vanishing. If the charges are vanishing, the duality equation does not arise, but also the auxiliary field does not appear in the Lagrangian any more.

$\frac{1}{12}G_{ACDE}G_B{}^{CDE}$. Inserting (3.33) and keeping in mind the Weyl rescaling (3.29) we find

$$\begin{aligned}\tilde{T}_{\mu\nu} = & \frac{1}{8}g_{\mu\nu}e^{-3\phi-\rho_3}(f_0 + (g_0 + c_{aI}\eta^{IJ}t_J^a)c_0 + (g_I^a + \frac{1}{2}\epsilon^{abc}T_{bI}^Jc_{cJ} + \frac{1}{2}t^{(ab)}c_{bI})\eta^{IK}c_{aK})^2 \\ & - \frac{1}{2}e^{-2\phi}g_{3ab}(D_\mu\gamma^a + \frac{1}{2}\epsilon^{acd}c_{cI}\eta^{IJ}D_\mu c_{dJ})(D_\nu\gamma^b + \frac{1}{2}\epsilon^{bef}c_{eK}\eta^{KL}D_\nu c_{fL}) \\ & + \frac{1}{2}e^{-2\phi}g_{\mu\nu}(D_\rho\gamma^a + \frac{1}{2}\epsilon^{acd}c_{cI}\eta^{IJ}D_\rho c_{dJ})(D^\rho\gamma^b + \frac{1}{2}\epsilon^{bef}c_{eK}\eta^{KL}D^\rho c_{fL}) \\ & + \frac{1}{2}e^{\rho_3}H^{IJ}F_{I\mu\rho}F_{J\nu}{}^\rho + \frac{1}{2}e^{-\phi+\rho_3}g^{3ab}\tilde{F}_{a\mu\rho}\tilde{F}_{b\nu}{}^\rho \\ & + \frac{1}{2}e^{-2\rho_3}D_\mu c_0 D_\nu c_0 - \frac{1}{2}e^{-\phi}H^{IJ}g_3^{ab}D_\mu c_{aI}D_\nu c_{bJ} ,\end{aligned}\tag{B.11}$$

$$\begin{aligned}\tilde{T}_{\mu a} = & \frac{1}{2}e^{-2\phi-\rho_3/3}(f_0 + (g_0 + c_{eI}\eta^{IJ}t_J^e)c_0 + (g_K^e + \frac{1}{2}\epsilon^{efg}T_{fK}^M c_{gM} + \frac{1}{2}t^{(ef)}c_{fK})\eta^{KL}c_{eL}) \\ & k_b^a(D_\mu\gamma^b + \frac{1}{2}\epsilon^{bcd}c_{cI}\eta^{IJ}D_\mu c_{dJ}) \\ & + \frac{1}{4}e^{2\rho_3/3-\phi}k_a^d\epsilon_{dbc}g_3^{cg}\tilde{F}_{g\mu\nu}(D^\nu\gamma^b + \frac{1}{2}\epsilon^{bef}c_{eI}\eta^{IJ}D^\nu c_{fJ}) \\ & + \frac{1}{2}e^{2\rho_3/3}(k^{-1})_a^b H^{IJ}F_{I\mu\nu}D^\nu c_{bJ} + \frac{1}{2}e^{-\rho_3/3}\epsilon_{\mu\nu\rho\sigma}(k^{-1})_a^b\tilde{F}_b^{\rho\sigma}D^\nu c_0 \\ & + \frac{1}{2}e^{-\phi-\rho_3/3}k_a^b\epsilon_{bcd}D_\mu c_{cI}H^{IJ}(g_J^d + c_0t_J^d + \epsilon^{def}T_{eJ}^K c_{fK} + t^{(de)}c_{eJ}) ,\end{aligned}\tag{B.12}$$

$$\begin{aligned}\tilde{T}_{ab} = & \frac{1}{2}e^{-\phi+\rho_3/3}g_{3ac}(D_\mu\gamma^c + \frac{1}{2}\epsilon^{cef}c_{eI}\eta^{IJ}D_\mu c_{fJ})g_{3bd}(D^\mu\gamma^d + \frac{1}{2}\epsilon^{dgh}c_{gK}\eta^{KL}D^\mu c_{hL}) \\ & + \frac{1}{4}e^{4\rho_3/3}(g_{3ab}g_3^{cd} - \delta_a^c\delta_b^d)\tilde{F}_c^{\mu\nu}\tilde{F}_{d\mu\nu} \\ & + \frac{1}{6}e^{\phi-5\rho_3/3}D_\mu c_0 D^\mu c_0\delta_{ab} + \frac{1}{2}e^{\rho_3/3}H^{IJ}D_\mu c_{aI}D^\mu c_{bJ} \\ & + \frac{1}{2}e^{-\phi-2\rho_3/3}H^{IJ}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd})(g_I^c + c_0t_I^c + \epsilon^{cef}T_{eI}^K c_{fK} + t^{(ce)}c_{eI}) \\ & (g_J^b + c_0t_J^b + \epsilon^{bgh}T_{gJ}^L c_{hL} + t^{(bg)}c_{gJ}) ,\end{aligned}\tag{B.13}$$

$$\begin{aligned}\tilde{T}_{\alpha\beta} = & \frac{1}{2}e^{-2\phi+4\rho_3/3}(g_0 + c_{aI}\eta^{IJ}t_J^a)^2 + \frac{1}{2}g_{3ab}e^{-\phi-2\rho_3/3}(H^{IJ}\delta_{\alpha\beta} - 2e^{\rho_4/2}\zeta^{\hat{a}(I}P_K^J)\omega_{\alpha\gamma}^K(I^{\hat{a}})_{\beta}^\gamma) \\ & (g_I^a + c_0t_I^a + \epsilon^{acd}T_{cI}^K c_{dK} + t^{(ab)}c_{bI})(g_J^b + c_0t_J^b + \epsilon^{bef}T_{eJ}^L c_{fL} + t^{(be)}c_{eJ}) \\ & + \frac{1}{2}e^{\rho_3/3}g_3^{ab}(H^{IJ}\delta_{\alpha\beta} - 2e^{\rho_4/2}\zeta^{\hat{a}(I}P_K^J)\omega_{\alpha\gamma}^K(I^{\hat{a}})_{\beta}^\gamma)D_\mu c_{aI}D^\mu c_{bJ} \\ & + \frac{1}{2}e^{\phi+4\rho_3/3}(H^{IJ}\delta_{\alpha\beta} - 2e^{\rho_4/2}\zeta^{\hat{a}(I}P_K^J)\omega_{\alpha\gamma}^K(I^{\hat{a}})_{\beta}^\gamma)F_I^{\mu\nu}F_{J\mu\nu} .\end{aligned}\tag{B.14}$$

Here we used (3.39) to replace \hat{F} , the scalar-tensor duality relation (3.47) to replace the tensor fields \hat{C}_a by their dual scalars γ^a and the electro-magnetic duality relation (3.46) to replace the magnetic vector fields C^a by their electric counterparts \tilde{C}_a .

C $N = 4$ gauged supergravity

We review here the basic notation of gauged $N = 4$ supergravity in the embedding tensor formalism [7, 8]. The theory consists of $n + 6$ electric and same number of magnetic vector fields $V^{\alpha M}$, where $\alpha = +, -$ denotes electric and magnetic components and $M = 1, \dots, n + 6$ labels the vector multiplet index. Here n denotes the number of vector multiplets of the theory. The magnetic vectors have no kinetic term and are therefore

auxiliary fields in the theory. The scalar split into one complex scalar $\tau = \tau_1 + i\tau_2$ in the gravity multiplet that specifies the $Sl(2)/SO(2)$ coset matrix

$$M_{\alpha\beta} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \quad (C.1)$$

The vector multiplet scalar fields combine into an $SO(6, n)/(SO(6) \times SO(n))$ coset matrix M_{MN} . The flat metric defining $SO(6, n)$ is denoted by η_{MN} . To complete the bosonic field content, there are also auxiliary two-form gauge fields $B^{MN} = B^{[MN]}$ and $B^{\alpha\beta} = B^{(\alpha\beta)}$.

The non-trivial data of $N = 4$ gauged supergravity are its charges, which are determined by $\xi_{\alpha M}$ and $f_{\alpha MNP} = f_{\alpha[MNP]}$. Then the following combinations of charges occur regularly

$$\begin{aligned} \Theta_{\alpha MNP} &= f_{\alpha MNP} - \xi_{\alpha[N}\eta_{P]M}, \\ \hat{f}_{\alpha MNP} &= f_{\alpha MNP} - \xi_{\alpha[M}\eta_{P]N} - \frac{3}{2}\xi_{\alpha N}\eta_{MP}. \end{aligned} \quad (C.2)$$

The scalar covariant derivatives are then given by

$$\begin{aligned} DM_{\alpha\beta} &= dM_{\alpha\beta} + \xi_{(\alpha|M|}M_{\beta)\gamma}V^{M\gamma} - \xi_{\rho M}\epsilon^{\rho\gamma}\epsilon_{\delta(\alpha}M_{\beta)\gamma}V^{M\delta}, \\ DM_{MN} &= dM_{MN} + 2V^P\alpha\Theta_{\alpha P(M}{}^QM_{N)Q}. \end{aligned} \quad (C.3)$$

The field strengths of the vector fields are given by

$$DV^{M\alpha} = dV^{M\alpha} - \frac{1}{2}\hat{f}_{\beta NP}{}^MV^{N\beta} \wedge V^{P\alpha} + \frac{1}{2}\epsilon^{\alpha\beta}\Theta_{\beta}{}^M{}_{NP}B^{NP} + \frac{1}{2}\xi_{\beta}^MB^{\alpha\beta}, \quad (C.4)$$

with $\epsilon^{+-} = 1$. The kinetic terms are then given by

$$\begin{aligned} S_{\text{kin}} &= \frac{1}{2\kappa_4^2} \int \left((*1)r - \frac{1}{4}DM_{MN} \wedge *DM^{MN} - \frac{1}{8}DM_{\alpha\beta} \wedge *DM^{\alpha\beta} \right. \\ &\quad \left. - 2\text{Im}(\tau)M_{MN}DV^{M+} \wedge *DV^{N+} + \text{Re}(\tau)\eta_{MN}DV^{M+} \wedge DV^{N+} \right), \end{aligned} \quad (C.5)$$

and the topological terms are given by

$$\begin{aligned} S_{\text{top}} &= \frac{1}{2\kappa_4^2} \int \left(\xi_{+M}\eta_{NP}V^{M-} \wedge V^{N+} \wedge dV^{P+} \right. \\ &\quad - (\hat{f}_{-MNP} + 2\xi_{-N}\eta_{MP})V^{M-} \wedge V^{N+} \wedge dV^{P-} \\ &\quad - \frac{1}{4}\hat{f}_{\alpha MNR}\hat{f}_{\beta PQ}{}^RV^{M\alpha} \wedge V^{N+} \wedge V^{P\beta} \wedge V^{Q-} \\ &\quad + \frac{1}{16}\Theta_{+MNP}\Theta_{-}{}^M{}_{QR}B^{NP} \wedge B^{QR} \\ &\quad \left. - \frac{1}{2}(\Theta_{-MNP}B^{NP} + \xi_{\alpha M}B^{+\alpha}) \wedge (dV^{M-} - \frac{1}{2}\hat{f}_{\alpha QR}{}^MV^{Q\alpha} \wedge V^{R-}) \right). \end{aligned} \quad (C.6)$$

The scalar potential of the theory is given by

$$\begin{aligned} S_{\text{pot}} &= \frac{1}{16\kappa_4^2} \int (*1) \left(f_{\alpha MNP}f_{\beta QRS}M^{\alpha\beta} \left(\frac{1}{3}M^{MQ}M^{NR}M^{PS} \right. \right. \\ &\quad \left. \left. + \left(\frac{2}{3}\eta^{MQ} - M^{MQ} \right) \eta^{NR}\eta^{PS} \right) \right. \\ &\quad \left. - \frac{4}{9}f_{\alpha MNP}f_{\beta QRS}\epsilon^{\alpha\beta}M^{MNPQRS} + 3\xi_{\alpha}^M\xi_{\beta}^NM^{\alpha\beta}M_{MN} \right), \end{aligned} \quad (C.7)$$

where we defined the totally antisymmetric tensor

$$M_{MNPQRS} = \epsilon^{mnpqrs} \nu_m^M \nu_n^N \nu_p^P \nu_q^Q \nu_r^R \nu_s^S , \quad (\text{C.8})$$

from the $SO(6, n)$ vielbein ν_m^M . Finally, the embedding tensor components obey a number of quadratic constraints that are necessary in order to ensure locality of the supergravity. These constraints are given by

$$\begin{aligned} \xi_\alpha^M \xi_{\beta M} &= 0 , \\ \xi_{(\alpha}^P f_{\beta) PMN} &= 0 , \\ 3f_{\alpha R[MN} f_{\beta PQ]}^R + 2\xi_{(\alpha[M} f_{\beta) NPQ]} &= 0 , \\ \epsilon^{\alpha\beta} \left(\xi_\alpha^P f_{\beta PMN} + \xi_{\alpha M} \xi_{\beta N} \right) &= 0 , \\ \epsilon^{\alpha\beta} \left(f_{\alpha MNR} f_{\beta PQ}^R - \xi_\alpha^R f_{\beta R[M[P} \eta_{Q]N]} - \xi_{\alpha[M} f_{\beta N][PQ]} + \xi_{\alpha[P} f_{\beta]Q}[MN] \right) &= 0 . \end{aligned} \quad (\text{C.9})$$

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